Thèse de doctorat
de l'Université Sorbonne Paris Cité
Préparée à l'Université Paris Diderot
École Doctorale de Sciences Mathématiques de Paris Centre ED 386
Laboratoire / Equipe de recherche: Topologie et Géométrie Algébriques

On quantum invariants: homological model for the coloured Jones polynomials and applications of quantum $sl(2|1)$

Par **Cristina Ana-Maria Palmer-Anghel**

Thèse de doctorat de Mathématiques

Dirigée par Professeur Christian Blanchet

Présentée et soutenue publiquement à Paris le 29 juin 2018
devant le jury composé de:

Présidente du jury		
Christine Lescop	Directrice de recherches CNRS	Université de Grenoble
Rapporteurs		
Jacob Rasmussen	Professeur	University of Cambridge
Emmanuel Wagner	Maître de conférences HDR	Université de Bourgogne
Examinateurs		
Catherine Gille	Maître de conférences	Université Paris Diderot
Rinat Kashaev	Professeur	Université de Genève
Directeur de thèse		
Christian Blanchet	Professeur des Universités	Université Paris Diderot

Institut de mathématiques de Jussieu-Paris Rive gauche.
UMR 7586.
Boîte courrier 247
4 place Jussieu
75252 Paris Cedex 05

Université Sorbonne Paris Cité.
Campus Paris Diderot.
École doctorale de sciences mathématiques de Paris centre.
Case courrier 7012
8 Place Aurélie Nemours
75205 Paris Cedex 13

A mon père,
pour m'avoir révélé la beauté des mathématiques.

If you can dream it,
you can do it.

Walt Disney,
communicated by Christian Blanchet

La découverte est le privilège de l'enfant.
C'est du petit enfant que je veux parler,
l'enfant qui n'a pas peur encore de se tromper,
d'avoir l'air idiot, de ne pas faire sérieux,
de ne pas faire comme tout le monde.

Alexandre Grothendieck

Piano Concerto No. 1 -Tchaikovsky

Remerciements

Tout d'abord je voudrais exprimer ma profonde gratitude à Professeur Christian Blanchet pour avoir accepté d'être mon directeur de thèse. Il a une intuition et une vision mathématique que j'admire. La question de trouver des modèles topologiques pour certaines invariantes quantiques m'a fascinée la première fois que je l'ai entendue, elle est restée avec moi pour chaque jour pendant ces trois années, et elle donne des très belles directions de recherche. Les discussions avec lui m'ont enrichie mathématiquement et j'ai été toujours heureuse d'entendre les questions très profondes qu'il a su révéler. Il m'a donnée une certaine liberté, mais dans le même temps il était disponible, et très encouragent, il a eu beaucoup de confiance en moi, especiallement dans plusieurs moments quand je n'ai pas trouvé ça dans moi-même. Je me rappelle clairement un jour de juillet quand il m'a dit le citât au-dessus- c'était un des plus beaux et plus difficiles moments dans ce travail. Pour tout ce que j'ai appris dans cette période, mathématiquement et au niveau humain je voudrais lui remercier.

I am grateful to and I would like to thank very much Nathan Geer. It has been a special chance to work with him on a collaboration and I have learned many things from him during these years. Especially in the first year of my PhD, while he was visiting Paris Diderot University, he was available and we had regular meetings. From the beginning he was very supportive and we have had many beautiful discussions.

I am very honoured to have Jacob Rasmussen as my referee and I would like to thank him very much for accepting it and reading very carefully this thesis.

Je suis très reconnaissante à Emmanuel Wagner et je voudrais lui remercier pour avoir accepté d'être mon rapporteur, lire le manuscrit soigneusement et pour discuter avec moi des remarques concernant ça.

Je voudrais remercier beaucoup à Catherine Gille, Rinat Kashaev et Christine Lescop pour avoir accepté de faire partie de mon jury.

I would like to thank to the organizers of the program "Homology Theories in Low Dimensional Topology" at the Isaac Newton Institute in Cambridge for welcoming me as a participant for two months in 2017. It was inspiring to be there. Especially I am thankful to Jacob Rasmussen for giving me the opportunity to talk there. The discussions with him after that talk were important to me.

Je suis reconnaissante à Emmanuel Wagner pour m'avoir invitée plusieurs fois à Dijon. On a eu beaucoup des discussions mathématiques et c'était toujours un grand plaisir d'être là. Je voudrais lui remercier de partager son savoir mathématique avec moi, son enthousiasme et pour beaucoup de choses que j'ai appris de lui.

J'aimerais également remercier la Région Ile-de-France pour financer mon program doctoral. Je suis très reconnaissante pour cette bourse et pour la possibilité de voyager beaucoup a plusieurs conferences, c'etait une période unique et spéciale dans mon parcours mathématique.

Je voudrais remercier à toutes mes professeurs qui j'ai entendu pendant mon Master 2 à l'Université Paris Diderot. J'ai eu l'opportunité d'avoir des très beaux courses de topologie dans petite dimension.

Mes remerciements vont à Anton Zorich. J'étais fortunée d'avoir des courses de M2 avec lui, et pendant ces années il était très encouragent et certaines avises de lui ont été spéciales pour moi.

Je voudrais remercier à tous les chercheurs de l'IMJ-PRG. Être dans ce cadre, avoir la possibilité de participer aux activités scientifiques, les Séminaires et Groupes de Travail c'était une très bonne experience. Spécialement, je suis reconnaissante à Gregor Masbaum, pour des discussions enrichissantes qu'on a eu récemment.

Je voudrais remercier à tous les responsables administratifs pour leur gentillesse et pour m'avoir aidée a me débrouiller dans chaque situation que j'ai rencontrer. Mes remerciements vont à Nadine Fournaiseau, Claire Lavollay, Élodie Destrebecq et Mailys Joligard.

Un grand merci aux doctorants d'IMJ-PRG, en particulier à Fathi, Felipe, Léo, Ramanujan, Louis pour leur amitié et pour l'ambiance au laboratoire. Je voudrais aussi remercier mes frères académiques, Marco et Daniel pour partager leur connaissances mathématiques avec moi, et pour les bonnes moments qu'on a eu ensemble. Je tiens à remercier Nicolina et Liana, pour leur amitié, qui a commencé à Bucarest quand on était plus jeune et a continué dans cette laboratoire. Je tiens aussi à dire un grand merci aux doctorants et chercheurs qui j'ai rencontré pendant les déplacements aux conferences, notamment à Jules, Céleste, Delphine, Gilberto, Ben, Paul, Filip. Un grand merci à Fathi pour m'introduire un peu dans le monde des invariants L^2 et pour des discussions enthousiastes qu'on a eu pendant les conferences dans les Pyrénées.

Merci beaucoup à Nicolina, je suis toujours très reconnaissante pour sa grande influence que je suis venue pour le M2 à Paris, me rappellant une très importante date limite.

Ensuite, c'est mon plaisir de remercier mes amis d'Université de Bucarest. Un grand merci à Miron et Alexandra pour une amitié constante et pour des beaux moments qu'on a eu au début de nos chemins mathématiques. Je suis très reconnaissante à mon amie Oana, pour être toujours proche, directe, encourageante, et pour partager la passion pour des bons cafés.

Je tiens à remercier à toutes les professeurs qui j'ai entendu à l'Université de Bucarest, pendant mes premières 4 années comme étudiante. Aussi, l'École Normale Supérieure de Bucarest (SNSB) était un très bon endroit où j'ai eu la chance d'avoir des cours très beaux, qui ont joué un rôle dans ce que je suis maintenant. Spécialement, je tiens à remercier à Dorin Cheptea, Sergiu Moroianu et Daniel Matei.

Je remercier profondément a mes parents pour leur amour et pour toute que je suis maintenant. C'est une occasion spéciale de dire merci à mon père- pour son modèle, pour son soutien continu pendant mon chemin mathématique, pour m'avoir révéler la beauté des mathématiques, depuis quand j'étais un enfant et on a eu des promenades sur le bord de la mer. Ii mulțumesc mult mamei mele, pentru ceea ce învăț de la ea în fiecare zi, pentru delicatețea ei, pentru momente muzicale speciale pe care le aduce printre noi, muzica de mai sus este pentru ea. Ensuite, un grand merci a ma soeur, Cipriana, pour être toujours proche, pour ton soutien, pour tes crêpes et bien sûr pour ton bon sens de l'humour. Un jour, j'espère que je vais apprendre la géométrie hyperbolique d'après toi.

Je tiens à remercier à ma famille, mes grands-parents Gheorghe și Sabina, Dragoș, Raluca, mes beaux parents- David and Renate et Sebastian. Je suis reconnaissante à mes grands-parents Felicia et Mircea, qui me manque beaucoup.

Enfin et surtout, merci à toi, Martin. Pour tout! Pour être comme tu est, pour ta noblesse. Tu me réponds toujours avec une gentilesse incroyable. Le deuxiemme citât au-dessus c'est pour toi. Un grand merci pour nos discussions concernant les représentations de Lawrence et pour ta richesse mathématique que tu as partagée très généreusement avec moi.

Vivre avec toi c'est vraiment un cadeau!

Résumé

Le domaine de cette thèse est dans la topologie quantique et son sujet est axé sur l'interaction entre la topologie de basse dimension et la théorie des représentations. Ma recherche concerne aspects différents des invariants quantiques pour les entrelacs et les 3-variétés, visant a créer des ponts entre les façons algébriques et topologiques de les définir. D'une part, une description algébrique et combinatoire pour un concept mathématique, crée l'opportunité de développer des outils de calcul. D'un autre côté, les descriptions topologiques et géométriques ouvrent des perspectives vers des constructions qui mènent a une compréhension plus profonde et a des théories plus subtiles.

Les polynômes de Jones coloriés sont des invariants quantiques d'entrelacs contruits en partant de la théorie des représentations de $U_q(sl(2))$. Le premier invariant de cette séquence est le polynôme de Jones original, qui peut-être caractérisé aussi par la théorie de l'écheveau. Bigelow et Lawrence ont décrit un modèle homologique pour le polynôme de Jones. Ils ont utilisé la représentation de Lawrence, qui est une représentation de groupe de tresses sur l'homologie des revêtements d'espaces de configurations dans le disque pointé, et la nature de l'écheveau de l'invariant pour la preuve. Contrairement a ce cas, les autres polynômes de Jones coloriés ne peuvent pas être définis facilement par la théorie de l'écheveau.

Dans la premiere partie de cette thèse, nous donnons un modèle topologique pour les polynômes de Jones coloriés. Nous utilisons leur définition comme invariants quantiques et construisons des correspondants topologiques pas à pas. Nous observons d'abord que l'invariant peut être codé par des espaces dits de plus haut poids, puis utiliser un résultat de Kohno, qui identifie ces espaces avec des représentations de Lawrence. Nous prouvons que les polynômes de Jones coloriés peuvent être obtenus comme une forme d'intersection géométrique gradués entre des classes d'homologie dans certaines couvertures des espaces de configuration de points dans le disque pointé.

Les deuxième et troisième parties sont orientées vers les applications de la théorie de la représentation des super groupes quantiques aux invariants quantiques. La deuxième partie est une collaboration avec N. Geer, ou nous construisons des invariants quantiques pour 3-variétés a partir des représentations de $U_q(sl(2|1))$. Turaev-Viro ont défini une méthode de type somme d'état qui donne des invariants de 3-variétés a partir de $U_q(sl(2))$.Pour les super groupes quantiques, cela entraîne l'annulation des invariants. Plus tard, Geer-Patureau-Turaev ont défini une méthode modifiée qui commence par une catégorie avec de bonnes propriétés et conduit à des invariants non-nulls. Notre stratégie consiste a construire une catégorie qui peut-être utilisée dans cette méthode modifiée. La troisième partie concerne l'étude des algèbre centralisatrices pour les représentations de $U_q(sl(2|1))$. Wagner et Marin conjecturaient les dimensions d'une suite d'algèbres centralisatrices correspondant à la représentation simple standard de $U_q(sl(2|1))$. Nous prouvons cette conjecture en utilisant des techniques combinatoires.

Mots-clés

Invariants quantiques, Noeuds, Polynômes de Jones coloriés, Super-algèbres

Abstract

The domain of this thesis is within quantum topology and its subject is focused towards the interaction between low dimensional topology and representation theory. My research concerns different aspects of quantum invariants for links and 3-manifolds, aiming to create bridges between algebraic and topological ways of defining them. On one hand, an algebraic and combinatorial description for a mathematical concept, creates the opportunity to develop computational tools. On the other hand, topological and geometrical descriptions open perspectives towards constructions that lead to a deeper understanding and more subtle theories.

The coloured Jones polynomials are quantum link invariants constructed from the representation theory of $U_q(sl(2))$. The first invariant of this sequence is the original Jones polynomial, which can be characterised also by skein theory. Bigelow and Lawrence described a homological model for the Jones polynomial. They used the Lawrence representation, which is a braid group representation on the homology of coverings of configuration spaces in the punctured disk, and the skein nature of the invariant for the proof. In contrast to this case, the other coloured Jones polynomials cannot be defined in an easy manner by skein theory.

In the first part of this thesis, we give a topological model for the coloured Jones polynomials. We use their definition as quantum invariants and construct step by step topological correspondents. We first observe that the invariant can be encoded through so-called highest weight spaces and then use a result by Kohno, which identifies these spaces with Lawrence representations. We prove that the coloured Jones polynomials can be obtained as graded geometric intersection pairings between homology classes in certain coverings of the configuration spaces of points in the punctured disk.

The second and third parts are oriented towards applications of representation theory of super quantum groups to quantum invariants. The second part is a collaboration with N. Geer, where we construct quantum invariants for 3-manifolds from representations of $U_q(sl(2|1))$. Turaev-Viro defined a state-sum type method that gives 3-manifold invariants from $U_q(sl(2))$. For super quantum groups, this leads to vanishing invariants. Later on, Geer-Patureau-Turaev defined a modified method which starts with a category with good properties and leads to non-vanishing invariants. Our strategy is to construct a category that fits into the input of this modified method.

The third part concerns the study of centralizer algebras for representations of $U_q(sl(2|1))$. Wagner and Marin conjectured the dimensions of a sequence of centralizer algebras corresponding to the simple standard $U_q(sl(2|1))$-representation. We prove this conjecture using combinatorial techniques.

Keywords

Quantum invariants, Knots, Coloured Jones polynomials, Super algebras

Contents

Introduction (In English)

The subject of this thesis is within the area of low dimensional topology, focused on the study of quantum invariants for links and 3-manifolds. It has three main research parts which are related to topological models for quantum invariants for links, constructions of quantum invariants for 3-manifolds from the representation theory of quantum groups at roots of unity and the study of centralizer algebras for representations of super quantum groups.

0.1 Quantum invariants - Historical context

Quantum invariants for links

The theory of quantum invariants started with the discovery of the celebrated Jones polynomial for knots and links in 1984. After that, Witten conjectured the existence of a generalisation of the Jones polynomial to an invariant for 3-manifolds. In 1989, Reshetikhin and Turaev proved this and introduced a method that having as input a quantum group leads to link invariants and 3-manifolds invariants. This construction is purely algebraic and combinatorial.

$$\begin{array}{cccc} & & \text{algebraic tools , quantum dimension} & \\ (\quad \mathscr{U} \quad , & V_1,...,V_n \quad) & \to & J_{V_1,...,V_n}(L,q) \\ \text{quantum group} & \in \mathscr{R}ep(\mathscr{U}) & \rightsquigarrow & \text{quantum link invariant} \end{array}$$

Since then, the theory of quantum invariants has become richer and richer and it has opened new perspectives in the study of link invariants.

The coloured Jones polynomials are a family of quantum invariants for links $\{J_N(L,q) | N \in \mathbb{N}^*\}$ introduced by V. Jones, constructed from the representation theory of the quantum group $U_q(sl(2))$. The definition of J_N fits into the Reshetikhin-Turaev construction, where the N^{th} dimensional representation V_N of the quantum group $U_q(sl(2))$ is used as the colour for the link components. The first invariant from this sequence, which corresponds to $N = 2$, is the original Jones polynomial.

During the last 20 years, many conjectures and results concerning the coloured Jones polynomials have been discovered. One research direction which has been widely developed was is the study of categorifications for link invariants, which are finer invariants. On this side, Khovanov introduced in 2000 a categorification for the Jones polynomial and later on he defined a categorification for the coloured Jones polynomials. The tools that he used are combinatorial and algebraic. On the other hand, there is a long-standing question which asks whether the Jones polynomial detects the unknot. In 2010, Kronheimer and Mrowka proved that the Khovanov homology detects the unknot. In 2012, Andersen showed that the coloured Jones polynomials detect the unknot.

Another research direction is related to the connections between quantum invariants and other geometrical invariants for knots and 3-manifolds. On this line, Bigelow and Lawrence constructed a homological model for the Jones polynomial. They presented this invariant as a graded intersection pairing of homology classes in a covering of a certain configuration space in the punctured disk.

Quantum invariants for 3-manifolds

At the same time as the development of the theory of quantum invariants for links, people introduced tools in order to define quantum invariants for 3-manifolds. Passing from the invariants for links towards invariants for 3-manifolds requires one to work with categories with stronger properties, more specifically, an essential requirement was to have a finite number of simple objects in the category and the semi-simplicity of the category. Many quantum groups with generic q do not have this property, having infinitely many simple representations. This is the reason for which the input data towards 3-manifolds invariants is often the representation theory of a quantum group at roots of unity.

Reshetikhin-Turaev in 1991-[77] developed a method that starting with any modular category leads to 3-manifold invariants. They presented a 3-manifold as a surgery along the link and using the Reshetikhin-Turaev link invariant they constructed the 3-manifold invariant. Implicitly, they used the notion of quantum dimension for objects. On the other hand, in 1992 [82], Turaev and Viro defined invariants for 3-manifolds using a finite number of representations from the representation theory of $U_q(sl(2))$ at roots of unity. Their approach is based on triangulations and they used the notion of $6j$-symbols and quantum dimensions for objects. The invariant is constructed in a state sum type formula.

The representation theory of a quantum group changes totally if we pass from generic q to a root of unity. In the case of $U_q(sl(2))$, we have the following correspondence (in the following formulas we denote $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$):

$$q \text{ generic} \qquad \mathscr{R}ep(U_q(sl(2))) \longleftrightarrow \mathbb{N}$$

$$V_N \longleftrightarrow \text{dimension } N \longleftrightarrow qdim = [N]_q$$

$$q = \xi_r = e^{\frac{2\pi i}{2r}} \quad \mathscr{R}ep(U_q(sl(2))) \longleftrightarrow \mathbb{C}$$

$$V_N \longleftrightarrow \text{dimension } N, \qquad N \in \{1, ..., r-1\} \longleftrightarrow qdim \simeq [N]_{\xi_r}$$

$$V_\lambda \longleftrightarrow \text{dimension } r, \lambda \in \mathbb{C} \setminus \{1, ..., r-1\} \longleftrightarrow qdim = [r]_{\xi_r} = 0$$

In their construction, they used the representations $\{V_1, .., V_{r-1}\}$ in a state sum type formula. In other words, they used the representations of the quantum group at roots of unity $U_\xi(sl(2))$ with non-vanishing quantum dimension. Later, people became interested in the study of super-quantum groups and the construction of quantum invariants using their representation theory. The issue that occurs a priori in this context is the fact that for super-quantum groups the quantum dimension is generically zero. This means that in this case, the corresponding Reshetikhin-Turaev and Turaev-Viro type invariants for links and 3-manifolds vanish.

In 2006, Geer and Patureau used super Lie algebras of type I and defined a method to replace the usual quantum dimension of an object with a "modified quantum dimension" in order to obtain non-vanishing link invariants. After that, Geer, Patureau and Turaev [33] described a conceptual method that introduces the notion of modified dimension for a more general category and leads to modified Reshetikhin-Turaev type link invariants. This showed that for super lie algebras of type I, the classical Reshetikhin-Turaev link invariants vanish while the Geer-Patureau-Turaev invariants are non-trivial. Pursuing this line, they introduced a tool [34] that leads to 3-manifold invariants having as input any so called "relative graded spherical category".

They replaced the usual quantum dimensions and $6j$-symbols with modified ones in a Turaev-Viro state-sum type construction.

Centraliser algebras

The notion of centraliser algebra is well-known in representation theory and refers to the study of intertwiner spaces corresponding to the tower of the tensor powers of certain representations. For example, if we fix H to be a Hopf algebra and $V \in \mathscr{R}ep(H)$, then the tensor powers of V have again a module structure over H. Furthermore, one can construct a tower of algebras, called "centraliser algebras"

$$C_n := End_H(V^{\otimes n})$$

A particular case is the situation where $H = \mathscr{U}(g)$ the enveloping algebra of a Lie algebra g or the quantisation $H_q = \mathscr{U}_q(g)$ namely the quantum enveloping algebra. For the case of the Lie algebra $sl(N)$ and the standard representation at the classical level $V \in \mathscr{R}ep(\mathscr{U}(sl(N)))$, one gets the following connection to the group algebra of the symmetric group S_n:

$$k[S_n] \twoheadrightarrow End_{\mathscr{U}(sl(N))}(V^{\otimes n})$$

which at the quantised level corresponds to the Hecke algebra:

$$H_n \twoheadrightarrow End_{\mathscr{U}_q(sl(N))}(V^{\otimes n}).$$

Wenzl studied [83] the standard representation V of $so(N)$ and its relation at the classical level to the Brauer algebra Br_n:

$$Br_n \twoheadrightarrow End_{\mathscr{U}(so(N))}(V^{\otimes n})$$

At the deformed level, Birman and Wenzl ([20]) showed that the quantisation of the Brauer algebra corresponds to the Birman-Murakami-Wenzl algebra:

$$BMW_n \twoheadrightarrow End_{\mathscr{U}_q(so(N))}(V^{\otimes n}).$$

The study of centraliser algebras for various representations of quantum groups has been broadly developed and has led to relations between different ways of describing invariants for links.

The Birman-Murakami-Wenzl alebras $\{BMW_n\}_{n\in\mathbb{N}}$ are a sequence of algebras which are defined as quotients of the group algebra of the braid group by cubic relations. Moreover, these algebras lead to the Kauffman invariant for links ([20], [68]).

Pursuing this line, the question of finding matrix unit bases for centraliser algebras has been studied for some important algebras which are related to quantum invariants for knots. On this subject Wenzl ([83]) and Ram and Wenzl ([74]) described a matrix unit basis for Brauer's centralizer algebras and for the Hecke algebras of type A. Moreover, in [12], Blanchet and Beliakova described a precise basis of matrix units for the Birman-Murakami-Wenzl algebra using idempotent elements and skein theory. In 2006, Lehrer and Zhang ([62]) studied the cases when the morphism obtained from the group algebra of the braid group onto the automorphism group of the tensor power of a certain representation, given by infinitesimal actions is surjective.

In the super quantum group setting, similar questions arise in connection with the Links-Gould invariant. This is a 2-variable link polynomial $LG(L;t_0,t_1) \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$ introduced in 1992 by Links and Gould [63], constructed from the representation theory of the super-quantum group $U_q(sl(2|1))$ and it is a renormalized type invariant for links. As we have seen, the Reshetikhin-Turaev method for constructing link invariants leads to vanishing polynomials if one uses as input a category of representations of a super-quantum group. The procedure of renormalization means to use the Reshetikhin-Turaev type construction evaluated on the link where one strand is cut, and correct this in a way that leads to a well defined invariant.

The Links-Gould invariant fits into the Geer-Patureau machinery that leads to renormalized invariants for links. More precisely it can be recovered by a certain specialisation of the Geer-Patureau invariant for the case of the super-quantum group $U_q(sl(2|1))$. Another interesting property is that the Links-Gould polynomial recovers by a specialisation the Alexander-Conway invariant for links. For the construction of $LG(L,t_0,t_1)$, a generic 4-dimensional representation $V(0,\alpha)$ of $U_q(sl(2|1))$ is used, which corresponds to a generic complex number $\alpha \in \mathbb{C}$. Marin and Wagner in [67] studied properties related to the sequence of centraliser algebras corresponding to this super-representation.

0.2 Main Results

Research program

The main question of my PhD was a research program with the aim of describing geometrical categorifications for certain quantum invariants using Floer-type methods. This program has two parts, each of them being a fundamental question on its own. Suppose that we are given a quantum invariant $I(L,q)$ that we are interested in studying.

Question 1: The first part is a topological project which aims to find a topological model for the quantum invariant I.

More precisely, topological in this context means to describe the quantum invariant as a graded intersection pairing between two homology classes represented by Lagrangian submanifolds in a certain covering of a configuration space.

Question 2: Once we have such a model, the second project is to pursue a graded Floer homology-type theory for the classes given by Lagrangians in order to obtain a geometrical categorification for the quantum invariant.

The main result of my thesis answers Question 1, describing a topological model for the coloured Jones polynomials.

I) Topological interpretations for quantum invariants

The coloured Jones polynomials $J_N(L,q)$ are a family of quantum invariants constructed from the representation theory of $U_q(sl(2))$ in an algebraic and combinatorial manner. We give a topological model for $J_N(L,q)$, describing it as a graded intersection pairing between two homology classes on a covering of the configuration space of the punctured disc.

Theorem. *(***Topological model for coloured Jones polynomials***)([6]) Consider the colour $N \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, there exist homology classes*

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_\gamma \quad \text{and} \quad \tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_\gamma$$

such that for any link L and $\beta_{2n} \in B_{2n}$ such that $L = \hat{\hat{\beta}}_{2n}^{or}$ (oriented plat closure), the N^{th} coloured Jones polynomial has the following topological expression:

$$J_N(L,q) = < \beta_{2n} \tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

(here γ and δ_{N-1} are certain specialisations of coefficients 1.7.1.2)

II) Modified Turaev-Viro Invariants

The second result of my thesis is a project joint with N. Geer, where we constructed examples of modified Turaev-Viro type quantum invariants for 3-manifolds from the representation theory of the super quantum group $U_q(sl(2|1))$ at roots of unity. We used the representation theory of $U_q(sl(2|1))$ in order to construct a relative $\mathbb{C}/\mathbb{Z}$-relative spherical category, which from the Geer-Patureau-Turaev machinery leads to 3-manifold quantum invariants.

Theorem. *(- , Geer)([8]) Let $\mathscr{C}$ be the category constructed using tensor powers of standard generic representations of $U_q(sl(2|1))$ at roots of unity. Consider $\mathscr{C}^N$ to be a certain purification of $\mathscr{C}$ with respect to some negligible morphisms. Then $\mathscr{C}^N$ is a $\mathbb{C}/\mathbb{Z}$-relative spherical category that leads to modified Turaev-Viro invariants for 3-manifolds.*

III) Centralizer algebras related to quantum $sl(2|1)$

The third direction of my thesis is related to the study of the centraliser algebras for the standard representation of $U_q(sl(2|1))$ and their relation with the braid groups and the Links Gould invariant. Let $\alpha \in \mathbb{C}\setminus\mathbb{Q}$ and $V(0,\alpha)$ be the corresponding 4-dimensional representation of the super quantum group $U_q(sl(2|1))$. The centraliser algebra corresponding to this representation is:

$$LG_n(\alpha) := Aut_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n})$$

In 2011, Marin and Wagner conjectured the dimension of this algebra. We proved this conjecture using combinatorial tools:

Theorem. *([7]) (Conjecture Marin-Wagner [67])*

$$dim(LG_{n+1}(\alpha)) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

0.3 Summary of the PhD contents

I) Topological model for coloured Jones polynomials

The main result of this thesis is a topological model for all coloured Jones polynomials. In this part, we will present a summary of the definitions and the main tools that we use in Section 1 in order to construct a topological model for $J_N(L,q)$.

The coloured Jones polynomials $\{J_N(L,q)\}_{N\in\mathbb{N}}$ are a family of quantum link invariants, constructed from the representation theory $\{V_N|N\in\mathbb{N}\}$ of the quantum group $U_q(sl(2))$. The N^{th} coloured Jones polynomial $J_N(L,q)$ is defined using a Reshetikhin-Turaev type construction.

In 1991, R. Lawrence ([59]) introduced a sequence of homological representations for the braid groups $\{H_{n,m}\}$ using the homology of a certain covering of the configuration space of m unordered points in the n-punctured disk. Using that, Bigelow and Lawrence ([16], [60]) constructed a homological interpretation for the original Jones polynomial. This invariant has many definitions, it is a quantum invariant, but it can be characterised also by skein relations. Their method for the proof uses the characterisation of the Jones polynomial using skein relations. For the coloured Jones polynomials, there are no skein relations that are easy to deal with. The strategy for our topological model for all colured Jones polynomials is to analyse at a deep level their definition as quantum invariants and to construct step by step a homological counterpart.

$(U_q(sl(2)),V_N)$	$\rightarrow$	Coloured Jones polynomial		Original Jones polynomial
(q generic, $N\in\mathbb{N}$)		$J_N(L,q)$	$\dashrightarrow^{N=2}$	$J_2(L,q)$
Description		Quantum inv / No skein		Quantum inv / Skein
Homological model		Theorem 1.7.0.1		Bigelow-Lawrence (2001)
Proof method		Quantum def		Skein Theory
Tools		$H_{2n,n(N-1)}$		$H_{2n,n}$

Issues and Ideas

The first remark in our proof is the fact that even if, a priori, $J_N(L,q)$ is constructed using the tensor power of the N-dimensional $U_q(sl(2))$-representation V_N, actually, the whole invariant can be seen passing through a particular so called highest weight space of this finite dimensional representation. Highest weight spaces are subspaces in a tensor power of quantum group representations, which are invariant under the braid group action.

Secondly, in 2012 Kohno showed ([57],[41]) a deep result that shows the fact that the highest weight spaces from the tensor power of representations of the $U_q(sl(2))$-Verma modules carry homological information. More precisely, he proved that these highest weight spaces are isomorphic to the homological Lawrence representations. Here there is a slight subtlety that we would like to discuss.

In the introduction of [42], it was mentioned that there are no homological models for the coloured Jones polynomials because the highest weight spaces of the tensor powers of finite dimensional $U_q(sl(2))$-modules do not yet have known homological interpretations.

Indeed, up to this moment, there are still no known topological models for the highest weight spaces from the tensor powers of V_N. However, we look at the inclusion from highest weight spaces of the finite dimensional module inside those for the Verma module. We remark that if the weight is bigger than the colour N, which is our case, this inclusion is strict. After that we show that in fact we can construct the coloured Jones polynomials, passing through these "bigger" highest weight spaces of the Verma module. Then, our method uses the Lawrence representation as a homological counterpart for these highest weight spaces.

A second technical issue that we would like to discuss here concerns the non-genericity of the parameters. There is a family of Verma modules $\hat{V}_\lambda$ for $U_q(sl(2))$ indexed by complex numbers $\lambda \in \mathbb{C}$. Kohno proved that for generic parameters $\lambda \in \mathbb{C}$, the braid group representations on the highest weight spaces of the Verma module $\hat{V}_\lambda$ are isomorphic to specialisations of Lawrence representations. For the proof, he passes through the monodromy of KZ connections and glues two very important theorems. Firstly, the Drinfeld-Kohno Theorem, which asserts that the braid group representations on the highest weight spaces from the Verma module at generic parameters are isomorphic to the monodromy of the corresponding KZ connection. Secondly, in 2012, for generic λ, Kohno identifies the braid group representations defined by the monodromy of the KZ connections with the braid group action on a certain specialisation of the corresponding Lawrence representation. The homological identification between the braid group action on highest weight spaces and the Lawrence representation is then proved for generic parameters. The natural numbers are clearly non-generic and the base for the KZ-representation used by Kohno for the two identifications explodes for these parameters.

The technical problem is that the coloured Jones polynomial $J_N(L,q)$ is encoded by non generic parameters, corresponding to $\lambda = N-1$. We saw that in order to construct J_N the representation $V_N \in Rep(U_q(sl(2)))$ is used. One can see easily that

$$V_N \subseteq \hat{V}_{N-1}$$

Then, to use our method we need to work with the highest weight spaces corresponding to non generic parameters. This is a subtle point and is related to the choice of the quantum group $U_q(sl(2))$ that we are working with. We use the quantum group over the ring $\mathbb{Z}[q^{\pm}, s^{\pm}]$ and a Verma module $\hat{V}$ that encodes all the other ones by the parameter s. Then, in order to arrive at the case that we are interested in, we need a specialisation $s = q^{\lambda}$. In this language, Kohno's Theorem asserts that the specialisation of the highest weight representation from the Verma module and the corresponding specialisation of the Lawrence representation are isomorphic. In [41], it was mentioned that the identification works for non-generic parameters as well. However, we discuss in detail the subtleties concerning this question in Section 1.5. The idea is that using this version of the quantum group, both quantum representations and Lawrence representations are actually specialisations of some representations over the Laurent polynomials. On the other hand, the KZ monodromy representation is not a specialisation of a representation over a ring of Laurent polynomials, and from here comes the issue with the specialisation at natural parameters.

Outline of the construction

We start with a link L and consider a braid $\beta_{2n} \in B_{2n}$ such that $L = \hat{\hat{\beta}}^{or}_{2n}$ (oriented plat closure). We analyse the link diagram at three levels: the cups (corresponding to the lower part of the plat closure), the braid in the middle and the caps from the upper part of the diagram. We use the Lawrence representation as a correspondent for the cups and a dual Lawrence representation to encode the cap level. The braid will correspond to the action of the braid group on the Lawrence representation. Finally, the coloured Jones polynomial, which corresponds to the evaluation of the Reshetikhin-Turaev algebraic method on the link will correspond to a graded intersection pairing between the Lawrence representation and its dual. Let us make this precise.

1)Firstly, we will study the meaning of the Reshetikhin-Turaev functor at the braid level β_{2n}.

We remark the fact that, having in mind that we have a closed link, not just the braid, the Resketikhin-Turaev construction at the braid level, which a priori uses $V_N^{\otimes 2n}$, actually passes through the so called highest weight spaces $W_{2n,n(N-1)} \subseteq V_N^{\otimes 2n}$. These highest weight spaces $W_{n,m} \subseteq V_N^{\otimes n}$ are very interesting and rich objects, and they carry representations of the braid group B_n, called quantum representations. There is not yet any known homological interpretation for these highest weight spaces $W_{n,m}$ of the finite dimensional representation $V_N^{\otimes n}$.

On the other hand, the highest weight spaces $\hat{W}_{n,m}$ of the Verma module at natural parameter $\hat{V}_{N-1}$ (an infinite dimensional module, that contains V_N) have a geometrical counterpart.

The result proved by Kohno in 2012 allows us to identify the quantum representation $\hat{W}_{n,m}$ with a certain specialisation of the Lawrence representation: $H_{n,m}|_{\psi_{N-1}}$ (the precise definition is given in the section 1.5.3). This creates a bridge between quantum representations, which are purely algebraic, and the homological Lawrence representations, which encode a richer geometrical structure. In our model, we use the Lawrence representation as a counterpart to the braid part of the diagram, and from Kohno's result, the action of the braid on the algebraic, respectively geometrical side correspond to one another.

2) We will translate the union of cups and caps on the geometric side, by describing a non-degenerate homological pairing. Actually, we will use $H_{2n,n(N-1)}$ to encode the algebraic co-evaluation (union of cups). For the evaluation (union of caps), we will use a "dual" Lawrence representation ([17]) $H^{\partial}_{n,m}$, which is a subspace of the homology relative to the boundary of the same covering of the configuration space. There is a sesquilinear pairing that relates these two dual spaces, called the Blanchfield pairing

$$<,>: H_{n,m} \otimes H^{\partial}_{n,m} \rightarrow \mathbb{Z}[x^{\pm}, d^{\pm}]$$

Let us denote the specialisation of the coefficients:

$$\alpha_{N-1} : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$$

$$\alpha_{N-1}(x) = q^{2(N-1)} \qquad \alpha_{N-1}(d) = -q^{-2}$$

Lemma 0.3.0.1. *Consider the specialised Blanchfield pairing:*

$$<,> |_{\alpha_{N-1}} : H_{n,m}|_{\alpha_{N-1}} \otimes H^{\partial}_{n,m}|_{\alpha_{N-1}} \to \mathbb{Q}(q)$$

This form is non-degenerate.

(The specialisation $|_{\alpha_{N-1}}$ means induction of representations along α_{N-1}.) The advantage of this non-degeneracy over a field, is that any element of the dual of the first space, can be described as the intersection with a fixed element in the second space. From this, using Kohno's correspondence, we translate the evaluation on $W_{n,m}$, as an element in $H^{\partial}_{2n,n(N-1)}$, and from the previous remark, it is obtained as a pairing $< \cdot, \mathscr{G} >$ with $\mathscr{G} \in H^{\partial}_{2n,n(N-1)}$.

3) Then we apply Kohno's theorem for the braid part, and use the Blanchfield pairing as a counterpart for the evaluation and co-evaluation. Putting it all together, we present a homological model for the coloured Jones polynomial $J_N(L,q)$ (1.6.0.1), where the homology classes are constructed using the specialisation α_{N-1}:

$$\mathscr{F}^N_n \in H_{2n,n(N-1)}|_{\alpha_{N-1}} \qquad \mathscr{G}^N_n \in H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$$

4) The last part is devoted to the construction of the homological model for $J_N(L,q)$, using homology classes that are not specialised. We show that if we increase the ring of coefficients to a field, by a specialisation that doesn't depend on the colour N, there exist two classes in the corresponding Lawrence representation which lead to $\mathscr{F}^N_n$ and $\mathscr{G}^N_n$ by the specialisation α_{N-1}. However, we still need to work over a field.

Firstly we consider the following specialisation:

$$\gamma : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(s,q)$$

$$\gamma(x) = s^2; \quad \gamma(d) = -q^{-2}.$$

Let us define the morphism:

$$\delta_{\lambda} : \mathbb{Q}(s,q) \to \mathbb{Q}(q)$$

$$\delta_\lambda(s) = q^\lambda$$

Then we get the following relation between these three specialisations:

$$\alpha_\lambda = \delta_\lambda \circ \gamma$$

We show that there exist two elements in the homology over the field with two variables:

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_\gamma \qquad \tilde{\mathscr{G}}_n^N \in H^\partial_{2n,n(N-1)}|_\gamma$$

such that they specialise to the previous classes:

$$\tilde{\mathscr{F}}_n^N|_{\delta_\lambda} = \mathscr{F}_n^N$$
$$\tilde{\mathscr{G}}_n^N|_{\delta_\lambda} = \mathscr{G}_n^N.$$

Using these classes and the previous topological model, we conclude that the N^{th} coloured Jones polynomial for L has following topological model (1.7.0.1):

$$J_N(L,q) = < \beta_{2n}\tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

The advantage of $\tilde{\mathscr{F}}_n^N$ and $\tilde{\mathscr{G}}_n^N$ is the fact that they live inside intrinsic Lawrence representations, constructed over the field with two variables $\mathbb{Q}(s,q)$ via γ and do not depend on α_{N-1}. We see the specialisation α_{N-1}, just after we specialise the Blanchfield pairing, in order to arrive at one variable.

II) Modified Turaev-Viro Invariants

The second direction of this thesis concerns the theory of 3-manifold quantum invariants from super quantum groups. In a collaboration with Nathan Geer, we have constructed 3-manifold invariants using a modified Turaev-Viro type construction ([82][34]) from the representation theory of the quantum group $U_q(sl(2|1))$ at roots of unity.

In 1992, Turaev and Viro defined a method that leads to invariants for links in 3-manifolds using the representation theory of $U_q(sl(2))$ at roots of unity, using the quantum dimension and $6j$-symbols for representations in a state-sum type construction. Similarly as for the link invariants, for super Lie algebras of type I, the associated quantum dimensions and the corresponding $6j$-symbols are zero which leads to invariants that vanish.

In 2011, Geer, Patureau and Turaev defined invariants for links in 3-manifolds from any so called relative spherical category. They introduced a modified quantum dimension and used the corresponding modified 6j-symbols in a state-sum type construction in order to obtain non-vanishing invariants. For the super-Lie algebras, the representation theory with q generic is already very rich, the simple modules being parametrised by a continuous family. In a state sum type construction, a finite number of simple objects is needed. For that, they required the category $\mathscr{C}$ to be graded by a group G, such that each piece $\mathscr{C}_g, g \in G$ has a finite number of simple objects. Another property from the classical case is the semi-simplicity of the category. They require $\mathscr{C}$ to be generically semi-simple, which means that except a small set $X \subseteq G$, any slice $\mathscr{C}_g$ for $g \in G$ is semi-simple.

We defined invariants for links in 3-manifolds using the representation theory of $U_q(sl(2|1))$ at roots of unity $q^l = 1$. The simple modules over $U_q(sl(2|1))$ with generic q are parametrised by $\mathbb{N} \times \mathbb{C}$. For the first component n small, the generic $U_q(sl(2|1))$ representation $V(n, \alpha)$ deforms to a representation at root of unity.

Firstly we consider $\mathscr{C}$ to be the tensor subcategory which is generated by retracts of tensor powers of modules of the type $V(0, \tilde{\alpha})$, for $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$, $\tilde{\alpha} \neq \frac{1}{4}(mod\ \mathbb{Z})$. After that, we proved that there exists a family of modified right traces on $\mathscr{C}$. Using the action of the quantum group, we have a $\mathbb{C}/\mathbb{Z}$-grading on this category, but for each piece there are a priori infinitely many simple objects, and the semi-simplicity is difficult to control outside the alcove. To overcome this, we considered $\mathscr{C}^N$ to be the quotient category of $\mathscr{C}$ by the negligible morphisms with respect to the modified right trace. Basically, we keep the same objects, but increase the isomorphism classes of objects, by identifying morphisms which differ by a negligible morphism. The effect is that summing with a module with vanishing modified dimension doesn't change the isomorphism class. The important point is that the modules on the edge of the alcove have vanishing modified dimension. We show that at the level of isomorphism classes of simple objects in $\mathscr{C}^N$, we keep just the modules $V(n, \tilde{\gamma})$ from $\mathscr{C}$ with $n \leq l$. Another main point concerns the semi-simplicity of the category. Finally, we prove that $\mathscr{C}^N$ is generically semi-simple. The fact that we avoided certain weights $\frac{1}{4}(mod\ \mathbb{Z})$, ensures that we can control the decomposition and the semi-simplicity of the tensor product for small natural components of the weights. Then, by an inductive argument, we can control all the semi-simplicity in the alcove, and once we hit its boundary, we can ignore the corresponding component thanks to the purification that we chose.

Theorem. *(-, Geer) The category $\mathscr{C}^N$ is a $\mathbb{C}/\mathbb{Z}$-relative spherical category that leads to modified Turaev-Viro invariants for 3-manifolds.*

III) Centralizer algebras related to quantum $sl(2|1)$

The third direction from my PhD is related to the study of centraliser algebras for $U_q(sl(2|1))$-representations. Concerning the representation theory of this super-quantum group, the finite dimensional and irreducible representations of $U_q(sl(2|1))$ with generic q are indexed by $\mathbb{N} \times \mathbb{C}$. As we have seen, the Links-Gould invariant is constructed using the simple 4-dimensional representation $V(0, \alpha)$, corresponding to a generic complex parameter $\alpha \in \mathbb{C}$. We study the sequence of centraliser algebras corresponding to the tensor powers of the representation $V(0, \alpha)$.

Consider the weight $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. Using the $R-$matrix of the algebra $U_q(sl(2|1))$, we obtain a Yang Baxter operator $R \in Aut_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes 2})$. In this way, we get a sequence of braid group representations:

$$\rho_n : B_n \to Aut_{U_q(sl(2|1))}\left(V(0,\alpha)^{\otimes n}\right) \qquad \rho_n(\sigma_i) = Id^{i-1} \otimes R \otimes Id^{n-i-1}$$

Using the tensor power of this representation, we obtain a sequence of centralizer algebras:

$$LG_n(\alpha) := End_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n})$$

In 2011, Marin and Wagner ([67]), proved that this morphism is surjective and studied the kernel for small n. Moreover, they showed that it factors through a cubic Hecke algebra denoted by $H(\alpha)$. Further on, they considered certain relations that are in the kernel of this map: r_2 for three strands and r_3 for the braid group with four strands. In this way, using these relations they defined a smaller quotient of the cubic Hecke algebra:

$$A_n(\alpha) := H_n(\alpha)/(r_2, r_3)$$

$$\begin{tikzcd} \rho_n : \mathbb{C}B_n \arrow[rr] \arrow[dr] & & LG_n(\alpha) \\ & A_n(\alpha) \arrow[ur] & \end{tikzcd}$$

We are interested to study properties related to the morphism ρ_n. Our motivation for this, is its close relation to the Links-Gould invariant for links. The study of the algebra $LG_n(\alpha)$ as well as the difference between this and $\mathbb{C}B_n$ is related to the local relations satisfied by the operator R. Also, they conjectured the dimension of the centralizer algebra $LG_n(\alpha)$:

Conjecture 1. *(Conjecture-Marin-Wagner [67])(-)*

$$dim(LG_{n+1}(\alpha)) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

They also conjectured that those relations (r_2 and r_3) are enough in order to generate the kernel of $\rho_n(\alpha)$. As a consequence one would get that:

Conjecture 2. *(Marin-Wagner [67])*

$$A_n(\alpha) \simeq LG_n(\alpha), \forall n \in \mathbb{N}$$

The main result of this third part of the thesis, is the proof of Conjecture 3. We use combinatorial techniques in order to encode the semi-simple decomposition of the tensor powers of of the canonical 4-dimensional $U_q(sl(2|1))$ representation. Firstly we pass from the initial algebraic question of computing the dimension of LG_n to a purely combinatorial problem. We construct certain diagrams in the lattice with integer coordinates on the plane, where each point has assigned a certain weight. This has the role of encoding the dimensions of the corresponding multiplicity spaces corresponding to $V(0,\alpha)^{\otimes n}$. Inductively, we obtain that the dimensions of these multiplicity spaces can be described using a way of counting paths in the plane with prescribed moves.

Our strategy starts with the semi-simple decomposition of $V(0,\alpha)^{\otimes 2}$. Moreover, for generic values of the parameter $\alpha \in \mathbb{C}$, $V(0,\alpha)^{\otimes n}$ is semi-simple. Then, we remark any automorphism of $V(0,\alpha)^{\otimes n}$ will decompose into blocks onto the isotypic components corresponding to the semi-simple decomposition of $V(0,\alpha)^{\otimes n}$. This shows that in order to compute the dimension of LG_n, it is enough to understand this semi-simple decomposition of the n^{th} tensor power of $V(0,\alpha)$.

For any $k \in \mathbb{N}$, we will encode the decomposition of $V(0,\alpha)^{\otimes k}$ into a diagram $D(k)$ in the plane. Each point $(x,y) \in \mathbb{N}\times\mathbb{N}$ will have a weight $T_k(x,y)$ in $D(k)$, which is given by the multiplicity of $V(x, k\alpha + y)$ inside $V(0,\alpha)^{\otimes k}$. After that, we encode the tensor with an additional $V(0,\alpha)$ combinatorially, at the level of diagrams. The conclusion is that $D(k+1)$ can be determined from $D(k)$, by making some local moves at each point, which we call "blow ups". Inductively, we obtain that each multiplicity $T_n(x,y)$ is actually the number of paths in the plane with length $n-1$ with certain allowed moves. The last part is related to a correspondence between this counting of paths, and another combinatorial problem of counting pairs of paths in the plane with some restrictions, for which the dimension was known. Putting this all together leads to the conjectured dimension.

Introduction (En Français)

Le sujet de cette these est dans le domaine de la topologie de petite dimension, centrée sur l'étude des invariants quantiques pour les entrelacs et les 3-variétés. Il a trois parties principales de recherche qui sont liées aux modeles topologiques pour les invariants quantiques pour les entrelacs, les constructions d'invariants quantiques pour les 3-variétés a partir de la théorie des représentations des groupes quantiques aux racines de l'unité et l'étude des algebres centralisatrices pour les représentations des groupes quantiques.

0.4 Invariants quantiques - Contexte historique

Invariants quantiques pour les entrelacs

La théorie des invariants quantiques a commencé avec la découverte du fameux polynôme de Jones pour les noeuds et les entrelacs en 1984. Apres cela, Witten conjectura l'existence d'une généralisation du polynôme de Jones a un invariant pour les 3-variétés. En 1989, Reshetikhin et Turaev l'ont prouvé et ont introduit une méthode qui a comme entrée un groupe quantique et qui conduit a des invariants d'entrelacs et des invariants des 3-variétés. Cette construction est purement algébrique et combinatoire.

$$\overset{\text{utils algebriques , dimension quantique}}{\left(\underset{\text{group quantique}}{\mathscr{U}} \quad , \quad \underset{\in \mathscr{R}ep(\mathscr{U})}{V_1,...,V_n} \right) \quad \underset{\rightsquigarrow}{\rightarrow} \quad \underset{\text{invariant quantique des entrelacs}}{J_{V_1,...,V_n}(L,q)}}$$

Depuis lors, la théorie des invariants quantiques est devenue plus riche et plus riche et elle a ouvert de nouvelles perspectives dans l'étude des invariants des entrelacs.

Les polynômes de Jones colorés sont une famille d'invariants quantiques pour les entrelacs $\{J_N(L,q)|N \in \mathbb{N}^*\}$ introduit par V. Jones, et construit a partir de la théorie de la représentation du groupe quantique $U_q(sl(2))$. La définition de J_N entre dans la construction de Reshetikhin-Turaev, ou la représentation N^{th} dimensionnelle V_N du groupe quantique $U_q(sl(2))$ est utilisée comme couleur pour les composants de l'entrelac. . Le premier invariant de cette suite, qui correspond a $N = 2$, est le polynôme original de Jones.

Au cours des 20 dernieres années, de nombreuses conjectures et résultats concernant les polynômes colorés de Jones ont été découverts. L'une des directions de recherche qui a été largement développée est l'étude des catégorisations pour les invariants des entrelacs, qui sont des invariants plus fins. De ce côté, Khovanov introduit en 2000 une catégorisation pour le polynôme de Jones et plus tard il définit une catégorification pour les polynômes de Jones colorés. Les outils qu'il a utilisés sont combinatoires et algébriques. D'un autre côté, il y a une question de longue date qui demande si le polynôme de Jones détecte le noeud trivial. En 2010, Kronheimer et Mrowka ont prouvé que l'homologie de Khovanov détecte le noeud trivial. En 2012, Andersen a montré que les polynômes colorés de Jones détectent aussi le noeud trivial.

Une autre direction de recherche est liée aux connexions entre les invariants quantiques et d'autres invariants géométriques pour les noeuds et les 3-variétés. Sur cette ligne, Bigelow et Lawrence ont construit un modele homologique pour le polynôme de Jones. Ils ont présenté cet invariant comme une intersection graduée de classes d'homologie dans un recouvrement d'un certain espace de configuration dans le disque pointee.

Invariants quantiques pour 3-variétés

En meme temps que le développement de la théorie des invariants quantiques pour les entrelacs, les gens ont introduit des outils afin de définir les invariants quantiques pour les 3-variétés. Passer des invariants pour les entrelacs vers les invariants pour les 3-variétés nécessite de travailler avec des catégories aux propriétés plus fortes, plus spécifiquement, une exigence essentielle était d'avoir un nombre fini d'objets simples dans la catégorie et la semi-simplicité de la catégorie. De nombreux groupes quantiques avec q générique n'ont pas cette propriété, ayant une infinité de représentations simples. C'est la raison pour laquelle la donnée d'entrée vers des invariants des 3-variétés est souvent la théorie de la représentation d'un groupe quantique aux racines de l'unité.

Reshetikhin-Turaev en 1991- [77] a développé une méthode qui, en commençant par n'importe quelle catégorie modulaire, conduit a des invariants des 3-variétés. Ils ont présenté une 3-variété comme une chirurgie le long d'un entrelac et en utilisant l'invariant de l'entrelac de Reshetikhin-Turaev, ils ont construit l'invariant de 3-variétés. Implicitement, ils ont utilisé la notion de dimension quantique pour les objets. D'autre part, en 1992 [82], Turaev et Viro ont défini des invariants pour les 3-variétés en utilisant un nombre fini de représentations de la théorie de la représentation de $U_q(sl(2))$ aux racines de l'unité. Leur approche est basée sur des triangulations et ils ont utilisé la notion de $6j$ - symboles et dimensions quantiques pour les objets. L'invariant est construit dans une formule de type somme d'état.

La théorie de représentation d'un groupe quantique change totalement si nous passons du q générique a une racine d'unité. Dans le cas de $U_q(sl(2))$, nous avons la correspondance suivante (dans les formules dessous, nous notons $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$):

$$q \text{ generic} \qquad \mathscr{R}ep(U_q(sl(2))) \longleftrightarrow \mathbb{N}$$

$$V_N \longleftrightarrow \text{dimension } N \longleftrightarrow qdim = [N]_q$$

$$q = \xi_r = e^{\frac{2\pi i}{2r}} \quad \mathscr{R}ep(U_q(sl(2))) \longleftrightarrow \mathbb{C}$$

$$V_N \longleftrightarrow \text{dimension } N, \qquad N \in \{1, ..., r-1\} \longleftrightarrow qdim \simeq [N]_{\xi_r}$$

$$V_\lambda \longleftrightarrow \text{dimension } r, \lambda \in \mathbb{C} \setminus \{1, ..., r-1\} \longleftrightarrow qdim = [r]_{\xi_r} = 0$$

Dans leur construction, ils ont utilisé les représentations $\{V_1, .., V_{r-1}\}$ dans une formule de type somme d'état. En d'autres termes, ils ont utilisé les représentations du groupe quantique aux racines de l'unité $U_\xi(sl(2))$ avec une dimension quantique non nulle. Plus tard, les gens se sont intéressés a l'étude des groupes super-quantiques et a la construction d'invariants quantiques en utilisant leur théorie des représentations. Le probleme qui se pose a priori dans ce contexte est le fait que pour les groupes super-quantiques, la dimension quantique est génériquement nulle. Cela signifie que dans ce cas, les invariants de type Reshetikhin-Turaev et Turaev-Viro correspondants pour les entrelacs et les 3-variétés sont nulls.

En 2006, Geer et Patureau ont utilisé des super algebres de Lie de type I et ont défini une méthode pour remplacer la dimension quantique habituelle d'un objet par une "dimension quantique modifiée" afin d'obtenir des invariants des entrelacs non nulles. Apres cela, Geer, Patureau et Turaev [33] ont décrit une méthode conceptuelle qui introduit la notion de dimension modifiée pour une catégorie plus générale et conduit a des invariants des entrelacs de type Reshetikhin-Turaev modifiés. Ceci a montré que pour les super-algebres de Lie de type I, les invariants des entrelacs de Reshetikhin-Turaev classiques sont nulls alors que les invariants de Geer-Patureau-Turaev sont non-triviaux. Poursuivant cette ligne, ils ont introduit un outil [34] qui conduit a des invariants des 3-variétés ayant comme entrée n'importe quelle "catégorie sphérique graduée relative". Ils ont remplacé les dimensions quantiques habituelles et les symboles $6j$ par des modifications dans une construction de type somme d'état Turaev-Viro.

Algebres centralisatrices

La notion d'algebre centralisatrice est bien connue dans la théorie des représentations et se réfere a l'étude des espaces d'entrelacement correspondant a la tour des puissances tensorielles de certaines représentations. Par exemple, si nous fixons H pour etre une algebre de Hopf et $V \in \mathscr{R}ep(H)$, alors les puissances tensorielles de V ont encore une structure de module sur H. De plus, on peut construire une tour d'algebres, appelée "algebres centralisatrices"

$$C_n := End_H(V^{\otimes n})$$

Un cas particulier est la situation ou $H = \mathscr{U}(g)$ l'algebre enveloppante d'une algebre de Lie g ou la quantification $H_q = \mathscr{U}_q(g)$ a savoir l'algebre enveloppante quantique. Pour le cas de l'algebre de Lie $sl(N)$ et la représentation standard au niveau classique $V \in \mathscr{R}ep(\mathscr{U}(sl(N)))$, on obtient la relation suivante avec l'algebre groupale du groupe symétrique S_n:

$$k[S_n] \twoheadrightarrow End_{\mathscr{U}(sl(N))}(V^{\otimes n})$$

qui au niveau quantifié correspond a l'algebre de Hecke:

$$H_n \twoheadrightarrow End_{\mathscr{U}_q(sl(N))}(V^{\otimes n}).$$

Wenzl a étudié [83] la représentation standard V de $so(N)$ et sa relation au niveau classique avec l'algebre de Brauer Br_n:

$$Br_n \twoheadrightarrow End_{\mathscr{U}(so(N))}(V^{\otimes n})$$

Au niveau déformé, Birman et Wenzl([20]) ont montré que la quantification de l'algebre de Brauer correspond a l'algebre de Birman-Murakami-Wenzl:

$$BMW_n \twoheadrightarrow End_{\mathscr{U}_q(so(N))}(V^{\otimes n}).$$

L'étude des algebres centralisées pour diverses représentations de groupes quantiques a été largement développée et a conduit a des relations entre différentes façons de décrire les invariants pour les entrelacs. Les algebres de Birman-Murakami-Wenzl $\{BMW_n\}_{n\in\mathbb{N}}$ sont une séquence d'algebres qui sont définies comme des quotients de l'algebre groupale du groupe de tresses par des relations cubiques. De plus, ces algebres conduisent a l'invariant de Kauffman pour les entrelacs ([20], [68]).

Poursuivant cette ligne, la question de trouver des bases d'unités matricielles pour les algebres centralisatrices a été étudiée pour quelques algebres importantes liées aux invariants quantiques des noeuds. Sur ce sujet Wenzl ([83]) et Ram et Wenzl ([74]) ont décrit une base matricielle pour les algebres centralisatrices de Brauer et pour les algebres de Hecke de type A. De plus, dans [12], Blanchet et Beliakova décrivent une base précise d'unités matricielles pour l'algebre de Birman-Murakami-Wenzl en utilisant des éléments idempotents et la théorie skein. En 2006, Lehrer et Zhang ([62]) ont étudié les cas où le morphisme obtenu de l'algèbre de groupe du groupe de tresses sur le groupe automorphisme du pouvoir tensoriel d'une certaine représentation donnée par des actions infinitésimales est surjectif.

Dans le contexte du groupe super-quantique, des questions similaires se posent en relation avec l'invariant Links-Gould. Ceci est un polynôme a deux variables $LG(L;t_0,t_1)\in\mathbb{Z}[t_0^{\pm1},t_1^{\pm1}]$ introduit en 1992 par Links et Gould [63], construit a partir de la théorie des représentations du groupe super-quantique $U_q(sl(2|1))$ et est un type d'invariant renormalisé pour les entrelacs. Comme nous l'avons vu, la méthode de Reshetikhin-Turaev pour construire des invariants des entrelacs conduit a des polynômes nulls si l'on utilise au depart une catégorie de représentations d'un groupe super-quantique. La procédure de renormalisation consiste a utiliser la construction de type Reshetikhin-Turaev évaluée sur l'entrelac ou un fil est coupé, et corriger ceci d'une maniere qui conduit a un invariant bien défini.

L'invariant de Links-Gould s'inscrit dans la machinerie de Geer-Patureau qui conduit a des invariants renormalisés pour les entrelacs. Plus précisément, il peut etre récupéré par une certaine spécialisation de l'invariant de Geer-Patureau dans le cas du groupe super-quantique $U_q(sl(2|1))$. Une autre propriété intéressante est que le polynôme Links-Gould récupere par une spécialisation l'invariant Alexander-Conway pour les entrelacs. Pour la construction de $LG(L, t_0, t_1)$, est utilisée une représentation 4-dimensionnelle générique $V(0,\alpha)$ de $U_q(sl(2|1))$, ce qui correspond a un nombre complexe générique $\alpha \in \mathbb{C}$. Marin et Wagner dans [67] ont étudié les propriétés liées a la séquence des algebres centralisatrices correspondant a cette super-représentation.

0.5 Résultats principaux

Programme de recherche

La principale question de mon doctorat était un programme de recherche visant a décrire les catégorisations géométriques de certains invariants quantiques utilisant des méthodes de type Floer. Ce programme comporte deux parties, chacune étant une question fondamentale en soi. Supposons qu'on nous donne un invariant quantique $I(L, q)$ que nous sommes intéressés a étudier.

Question 1: La premiere partie est un projet topologique qui vise a trouver un modele topologique pour l'invariant quantique I.

Plus précisément, topologique dans ce contexte signifie décrire l'invariant quantique comme un intersection gradué entre deux classes d'homologie représentées par des sous-variétés lagrangiennes dans une certaine recouvrement d'un espace de configuration.

Question 2: Une fois que nous avons un tel modele, le second projet consiste a poursuivre une théorie de type d'homologie de Floer gradué pour les classes données par les lagrangiens afin d'obtenir une catégorification géométrique pour l'invariant quantique.

Le résultat principal de ma these répond a la question 1, décrivant un modele topologique pour les polynômes de Jones colorés.

I) Interprétations topologiques pour les invariants quantiques

Les polynômes de Jones colorés $J_N(L, q)$ sont une famille d'invariants quantiques construits a partir de la théorie de la représentation de $U_q(sl(2))$ de maniere algébrique et combinatoire. Nous donnons un model topologique pour $J_N(L, q)$, en le décrivant comme un intersection gradué entre deux classes d'homologie sur un recouvrement de l'espace de configuration du disque pointee.

Theorem. *(***Modele topologique pour les polynômes de Jones colorés***)([6]) Soit la couleur $N \in \mathbb{N}$. Alora, pour tout $n \in \mathbb{N}$, ils existent des classes d'homologie*

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_{\gamma} \quad \textit{and} \quad \tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\gamma}$$

tel que pour tout entrelac L et $\beta_{2n} \in B_{2n}$ tel que $L = \hat{\hat{\beta}}_{2n}^{or}$ (fermeture platte orientée), le N^{th}-polynôme de Jones coloré a l'expression topologique suivante:

$$J_N(L,q) =< \beta_{2n}\tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

(ici γ et δ_{N-1} sont certaines spécialisations de coefficients 1.7.1.2)

II) Invariants de Turaev-Viro modifiés

Le second résultat de ma these est un projet en colaboration avec N. Geer, ou nous avons construit des exemples d'invariants quantiques de type Turaev-Viro modifiés pour 3-variétés a partir de la théorie de représentation du groupe super quantique $U_q(sl(2|1))$ aux racines de l'unité. Nous avons utilisé la théorie de la représentation de $U_q(sl(2|1))$ pour construire une catégorie sphérique relative $\mathbb{C}/\mathbb{Z}$, qui, de la machinerie de Geer-Patureau-Turaev conduit a des invariants quantiques pour les 3-varietees.

Theorem. *(- , Geer)([8]) Soit $\mathscr{C}$ la catégorie construite en utilisant les puissances tensorielles des représentations génériques standard de $U_q(sl(2|1))$ aux racines de l'unité. Soit $\mathscr{C}^N$ une certaine purification de $\mathscr{C}$ par rapport aux certains morphismes négligeables. Alors $\mathscr{C}^N$ est une $\mathbb{C}/\mathbb{Z}$-catégorie sphérique relative qui conduit a des invariants Turaev-Viro modifiés pour 3-variétés.*

III) Algebres centralisatrices liées au $sl(2|1)$ quantique

La troisieme direction de ma these est liée a l'étude des algebres centralisatrices pour la représentation standard de $U_q(sl(2|1))$ et leur relation avec les groupes de tresses et l'invariant Links Gould. Soit $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ et $V(0,\alpha)$ la représentation 4 -dimensionnelle correspondante du super-group quantique

$U_q(sl(2|1))$. L'algebre centralisatrice correspondant a cette représentation est:

$$LG_n(\alpha) := Aut_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n})$$

En 2011, Marin et Wagner ont conjecturé la dimension de cette algebre. Nous avons démontré cette conjecture en utilisant des outils combinatoires:

Theorem. *([7]) (Conjecture Marin-Wagner [67])*

$$dim(LG_{n+1}(\alpha)) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

0.6 Résumé du contenu de la these

I) Modele topologique pour les polynômes de Jones colorés

Le résultat principal de cette these est un modele topologique pour tous les polynômes de Jones colorés. Dans cette partie, nous allons présenter un résumé des définitions et des principaux outils que nous utilisons dans la Section 1 afin de construire un modele topologique pour $J_N(L,q)$.

Les polynômes de Jones colorés $\{J_N(L,q)\}_{N\in\mathbb{N}}$ sont une famille d'invariants quantiques pour les entrelacs , construits a partir de la théorie des représentations $\{V_N|N \in \mathbb{N}\}$ du groupe quantique $U_q(sl(2))$. Le N^{th} polynôme de Jones coloré $J_N(L,q)$ est défini en utilisant une construction de type Reshetikhin-Turaev.

En 1991, R. Lawrence a introduit une séquence de représentations homologiques pour les groupes de tresses $\{H_{n,m}\}$ en utilisant l'homologie d'un certain recouvrement de l'espace de configuration de m de points non ordonnés dans le disque n-pointee. En utilisant cela, Bigelow et Lawrence ([16], [60]) construisirent une interprétation homologique pour le polynôme original de Jones. Cet invariant a de nombreuses définitions, c'est un invariant quantique, mais il peut aussi etre caractérisé par des relations de skein. Leur méthode pour la preuve utilise la caractérisation du polynôme de Jones en utilisant des relations de skein. Pour les polynômes de Jones colorés, il n'y a pas de relations skein faciles a gérer. La stratégie pour notre modele topologique pour tous les polynômes de Jones colorés est d'analyser a un niveau profond leur définition en tant qu'invariants quantiques et de construire, étape par étape, une contrepartie homologique.

$(U_q(sl(2)), V_N)$	$\rightarrow$	Polynome Jones colorie		Polynome Jones original
(q generique, $N \in \mathbb{N}$)		$J_N(L,q)$	$\dashrightarrow^{N=2}$	$J_2(L,q)$
Description		Inv quantique / Non skein		Inv quantique / Skein
Model homologique		Theorem 1.7.0.1		Bigelow-Lawrence (2001)
Methode de dem.		Def quantique		Theorie Skein
Utils		$H_{2n,n(N-1)}$		$H_{2n,n}$

Problemes et idées

La premiere remarque dans notre démonstration est le fait que meme si, a priori, $J_N(L,q)$ est construit en utilisant la puissance tensorielle de N-dimensional $U_q(sl(2))$-représentation V_N, en fait, tout l'invariant peut etre vu passant a travers un espace dit de poids le plus élevé de cette représentation de dimension finie. Les espaces de poids les plus élevés sont des sous-espaces dans une puissance tensorielle de représentations de groupes quantiques, qui sont invariantes sous l'action de groupe de tresses.

Deuxiemement, en 2012 Kohno a montré ([57], [41]) un résultat profond qui montre le fait que les espaces de poids les plus élevés du pouvoir tensoriel des représentations de $U_q(sl(2))$-modules de Verma portent des informations homologiques. Plus précisément, il a prouvé que ces espaces de poids les plus élevés sont isomorphes aux représentations homologiques de Lawrence. Ici, il y a une légere subtilité dont nous aimerions discuter.

Dans l'introduction de [41], il a été mentionné qu'il n'y a pas de modeles homologiques pour les polynômes de Jones colorés car les espaces de poids les plus élevés des puissances tensorielles des $U_q(sl(2))$-modules finis n'ont pas des interprétations homologiques conues.

En effet, jusqu'a ce jour, il n'existe toujours pas de modeles topologiques connus pour les espaces de poids les plus élevés a partir des puissances tensorielles de V_N. Cependant, nous considérons l'inclusion des espaces de poids les plus élevés du module de dimension finie a l'intérieur de ceux du module Verma. Nous remarquons que si le poid est plus grand que la couleur N, ce qui est notre cas, cette inclusion est stricte. Apres cela, nous montrons qu'en fait nous pouvons construire les polynômes de Jones colorés, en passant par ces "plus grands" espaces de poids les plus élevés du module de Verma. Ensuite, notre méthode utilise la représentation de Lawrence comme contrepartie homologue pour ces espaces de poids les plus élevés.

Un deuxieme probleme technique dont nous aimerions discuter ici concerne la non-généricité des parametres. Il existe une famille de modules Verma $\hat{V}_\lambda$ pour $U_q(sl(2))$ indexée par des nombres complexes $\lambda \in \mathbb{C}$. Kohno a prouvé que pour les parametres génériques $\lambda \in \mathbb{C}$, les représentations de groupes de tresses sur les espaces de poids les plus élevés du module Verma $\hat{V}_\lambda$ sont isomorphes aux spécialisations des représentations de Lawrence. Pour la preuve, il passe par la monodromie des connexions KZ et colle deux théoremes tres importants. Premierement, le théoreme de Drinfeld-Kohno, qui affirme que les représentations de groupes de tresses sur les espaces de poids les plus élevés du module de Verma aux parametres génériques sont isomorphes a la monodromie de la connexion KZ correspondante. Deuxiemement, en 2012, pour le λ générique, Kohno identifie les représentations de groupes de tresses définies par la monodromie des connexions KZ avec l'action de groupe de tresses sur une certaine spécialisation de la représentation de Lawrence correspondante. L'identification homologique entre l'action du groupe de tresses sur les espaces de poids les plus élevés et la représentation de Lawrence est ensuite prouvée pour les parametres génériques. Les nombres naturels sont clairement non génériques et la base de la représentation KZ utilisée par Kohno pour les deux identifications explose pour ces parametres.

Le probleme technique est que le polynôme coloré de Jones $J_N(L,q)$ est codé par des parametres non génériques, correspondant a $\lambda = N-1$. Nous avons vu que pour construire J_N est utilisée la représentation $V_N \in Rep(U_q(sl(2)))$. On peut facilement voir que

$$V_N \subseteq \hat{V}_{N-1}$$

Ensuite, pour utiliser notre méthode, nous devons travailler avec les espaces de poids les plus élevés correspondant a des parametres non génériques. C'est un point subtil ici, lié au choix du groupe quantique $U_q(sl(2))$ avec lequel nous travaillons. Nous utilisons le groupe quantique sur l'anneau $\mathbb{Z}[q^{\pm}, s^{\pm}]$ et un module Verma $\hat{V}$ qui encode tous les autres par le parametre

s. Ensuite, pour arriver au cas qui nous intéresse, nous avons besoin d'une spécialisation $s = q^{\lambda}$. Dans ce langage, le théoreme de Kohno affirme que la spécialisation de la représentation du poids le plus élevé du module de Verma et la spécialisation correspondante de la représentation de Lawrence sont isomorphes. Dans [41], il a été mentionné que l'identification fonctionne également pour les parametres non génériques. Cependant, nous discutons en détail des subtilités concernant cette question dans la section 1.5. L'idée est qu'en utilisant cette version du groupe quantique, les représentations quantiques et les représentations de Lawrence sont en fait des spécialisations de certaines représentations sur les polynômes de Laurent. D'autre part, la représentation de la monodromie KZ n'est pas une spécialisation d'une représentation sur un anneau de polynômes de Laurent, et d'ou le probleme de la spécialisation aux parametres naturels.

Esquisse de la construction

Nous commençons par un entrelac L et considérons une tresse $\beta_{2n} \in B_{2n}$ telle que $L = \hat{\hat{\beta}}_{2n}^{or}$ (fermeture plate orientée). Nous analysons le diagramme de l'entrelac a trois niveaux: les "cups" (correspondant a la partie inférieure de la fermeture plate), la tresse au milieu et les "caps" de la partie supérieure du diagramme. Nous utilisons la représentation de Lawrence comme correspondant pour les "cups" et une représentation de Lawrence douale pour encoder le niveau des "caps". La tresse correspondra a l'action du groupe de tresses sur la représentation de Lawrence. Enfin, le polynôme de Jones coloré, qui correspond a l'évaluation par la méthode algébrique de Reshetikhin-Turaev sur l'entrelac, correspond a un intersection gradué entre la représentation de Lawrence et son dual. Faisons-le précis.

1)Premierement, nous étudierons la signification du foncteur Reshetikhin-Turaev au niveau de la tresse β_{2n}.

Nous remarquons que, compte tenu du fait que nous avons un entrelac fermé, pas seulement la tresse, la construction de Resketikhin-Turaev au niveau de la tresse, qui utilise a priori $V_N^{\otimes 2n}$, passe en fait par ce que l'on appelle les espaces de poids les plus élevés $W_{2n,n(N-1)} \subseteq V_N^{\otimes 2n}$. Ces espaces de poids les plus élevés $W_{n,m} \subseteq V_N^{\otimes n}$ sont des objets tres intéressants et riches, et ils portent des représentations du groupe de tresses B_n, appelées représentations quantiques. Il n'y a pas encore d'interprétation homologique connue pour ces espaces de poids les plus élevés $W_{n,m}$ de la représentation en dimension finie $V_N^{\otimes n}$.

D'autre part, les espaces de poids les plus élevés $\hat{W}_{n,m}$ du module Verma au parametre naturel $\hat{V}_{N-1}$ (un module de dimension infinie, qui contient V_N) ont une contrepartie géométrique.

Le résultat prouvé par Kohno en 2012 nous permet d'identifier la représentation quantique $\hat{W}_{n,m}$ avec une certaine spécialisation de la représentation de Lawrence: $H_{n,m}|_{\psi_{N-1}}$ (la définition précise est donnée dans la section 1.5.3). Cela crée un pont entre les représentations quantiques, qui sont purement algébriques, et les représentations homologiques de Lawrence, qui codent une structure géométrique plus riche. Dans notre modele, nous utilisons la représentation de Lawrence comme contrepartie de la partie tressée du diagramme, et d'apres le résultat de Kohno, l'action de la tresse sur le côté algébrique, respectivement géométrique, correspond l'une a l'autre.

2) Nous traduirons l'union des "cups" et des "caps" du côté géométrique, en décrivant un couplage homologique non dégénéré. En fait, nous utiliserons $H_{2n,n(N-1)}$ pour coder la co-évaluation algébrique (union des "cups"). Pour l'évaluation (union des "caps"), nous utiliserons une représentation "dual" de Lawrence ([17]) $H^{\partial}_{n,m}$, qui est un sous-espace de l'homologie relative a la frontiere du meme recouvrement de l'espace de configuration. Il y a un couplage sesquilinéaire qui relie ces deux espaces douales, appelé le couplage de Blanchfield

$$<,>: H_{n,m} \otimes H^{\partial}_{n,m} \to \mathbb{Z}[x^{\pm}, d^{\pm}]$$

Notons la spécialisation des coefficients:

$$\alpha_{N-1} : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$$

$$\alpha_{N-1}(x) = q^{2(N-1)} \qquad \alpha_{N-1}(d) = -q^{-2}$$

Lemma 0.6.0.1. *Considérez le couplage spécialisé de Blanchfield:*

$$<,> |_{\alpha_{N-1}} : H_{n,m}|_{\alpha_{N-1}} \otimes H^{\partial}_{n,m}|_{\alpha_{N-1}} \to \mathbb{Q}(q)$$

Cette forme est non-dégénérée.

(La spécialisation $|_{\alpha_{N-1}}$ signifie l'induction de représentations le long de α_{N-1}.)
L'avantage de cette non-dégénérescence sur un corp, est que tout élément du dual du premier espace peut etre décrit comme l'intersection avec un élément fixe dans le second espace. A partir de cela, en utilisant la correspondance de Kohno, nous traduisons l'évaluation sur $W_{n,m}$, en tant qu'élément de

$H^{\partial}_{2n,n(N-1)}$, et de la remarque précédente, il est obtenu comme un couplage $< \cdot, \mathscr{G} >$ avec $\mathscr{G} \in H^{\partial}_{2n,n(N-1)}$.

3)Ensuite, nous appliquons le théoreme de Kohno pour la partie tresse, et nous utilisons le couplage de Blanchfield comme contrepartie pour l'évaluation et la co-évaluation. En mettant tout cela ensemble, nous présentons un modele homologique pour le polynôme de Jones coloré $J_N(L,q)$ (1.6.0.1), ou les classes d'homologie sont construites en utilisant la spécialisation α_{N-1}:

$$\mathscr{F}_n^N \in H_{2n,n(N-1)}|_{\alpha_{N-1}} \qquad \mathscr{G}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$$

4) La derniere partie est consacrée a la construction du modele homologique $J_N(L,q)$, en utilisant des classes d'homologie qui ne sont pas spécialisées. Nous montrons que si nous augmentons l'anneau de coefficients a un corp, par une spécialisation qui ne dépend pas de la couleur N, il existe deux classes dans la représentation de Lawrence correspondante qui conduisent a $\mathscr{F}_n^N$ et $\mathscr{G}_n^N$ par la spécialisation α_{N-1}. Cependant, nous devons encore travailler sur un corp.

Premierement, nous considérons la spécialisation suivante:

$$\gamma : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(s,q)$$
$$\gamma(x) = s^2; \quad \gamma(d) = -q^{-2}.$$

Allons définir le morphisme:

$$\delta_\lambda : \mathbb{Q}(s,q) \to \mathbb{Q}(q)$$
$$\delta_\lambda(s) = q^\lambda$$

Ensuite, nous obtenons la relation suivante entre ces trois spécialisations:

$$\alpha_\lambda = \delta_\lambda \circ \gamma$$

Nous montrons qu'il existe deux éléments dans l'homologie sur le corp avec deux variables:

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_{\gamma} \qquad \tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\gamma}$$

tels qu'ils se spécialisent dans les classes précédentes:

$$\tilde{\mathscr{F}}_n^N|_{\delta_\lambda} = \mathscr{F}_n^N$$
$$\tilde{\mathscr{G}}_n^N|_{\delta_\lambda} = \mathscr{G}_n^N.$$

En utilisant ces classes et le modele topologique précédent, nous concluons que le N^{th} polynome de Jones coloré pour L a le modele topologique suivant (1.7.0.1):

$$J_N(L,q) =< \beta_{2n}\tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

L'avantage de $\tilde{\mathscr{F}}_n^N$ et $\tilde{\mathscr{G}}_n^N$ est le fait qu'ils vivent a l'intérieur des représentations intrinseques de Lawrence, construites sur le corp avec deux variables $\mathbb{Q}(s,q)$ via γ et ne dépend pas de α_{N-1}. Nous voyons la spécialisation α_{N-1}, juste apres nous spécialisons le couplage de Blanchfield, afin d'arriver a une variable.

II) Invariants de Turaev-Viro modifiés

La deuxieme direction de cette these concerne la théorie des invariants quantiques des trois variétés obtenus des groupes super-quantiques. Dans une collaboration avec Nathan Geer, nous avons construit des invariants des 3-variétés en utilisant une construction de type Turaev-Viro modifiée ([82][34]) a partir de la théorie de la représentation du groupe quantique $U_q(sl(2|1))$ aux racines de l'unité.

En 1992, Turaev et Viro ont défini une méthode qui conduit a des invariants pour les entrelacs dans les 3-variétés en utilisant la théorie des représentations de $U_q(sl(2))$ aux racines de l'unité, en utilisant la dimension quantique et $6j$-symbols dans une construction de type somme d'état. De meme que pour les invariants des entrelacs, pour les super algebres de Lie de type I, les dimensions quantiques associées et les symboles $6j$ correspondants sont nuls, ce qui conduit a des invariants qui s'annulent.

En 2011, Geer, Patureau et Turaev ont défini des invariants pour les entrelacs dans les 3-variétés a partir de toute catégorie dite sphérique relative. Ils ont introduit une dimension quantique modifiée et ont utilisé les symboles 6j modifiés correspondants dans une construction de type somme d'état afin d'obtenir des invariants non nulls. Pour les super-algebres de Lie, la théorie des représentations avec q generique est déja tres riche, les modules simples étant paramétrés par une famille continue. Dans une construction de type somme d'état, un nombre fini d'objets simples est nécessaire. Pour cela, ils ont demandé que la catégorie $\mathscr{C}$ soit graduee par un groupe G, de sorte que chaque partie $\mathscr{C}_g, g \in G$ ait un nombre fini d'objets simples. Une autre propriété du cas classique est la semi-simplicité de la catégorie. Ils requierent que $\mathscr{C}$ soit génériquement semi-simple, ce qui signifie qu'a l'exception d'un petit ensemble $X \subseteq G$, toute tranche $\mathscr{C}_g$ pour $g \in G$ est semi-simple.

Nous avons défini les invariants pour les entrelacs dans les 3-variétés en utilisant la théorie des représentations de $U_q(sl(2|1))$ aux racines de l'unité $q^l = 1$. Les modules simples sur $U_q(sl(2|1))$ avec q générique sont paramétrés par $\mathbb{N} \times \mathbb{C}$. Pour le premier composant n petit, la représentation générique $V(n, \alpha)$ de $U_q(sl(2|1))$ se déforme en une représentation a la racine de l'unité.

Premierement nous considérons que $\mathscr{C}$ est la sous-catégorie tensorielle qui est générée par des rétractions de puissances tensorielles de modules de type $V(0, \tilde{\alpha})$, pour $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$, $\tilde{\alpha} \neq \frac{1}{4}(mod\ \mathbb{Z})$. Apres cela, nous avons prouvé qu'il existe une famille de traces droites modifiées sur $\mathscr{C}$. En utilisant l'action du groupe quantique, nous avons un $\mathbb{C}/\mathbb{Z}$-graduation sur cette catégorie, mais pour chaque morceau il y a a priori un nombre infini d'objets simples, et la semi-simplicité est difficile a contrôler en dehors du alcôve. Pour surmonter cela, nous avons considéré que $\mathscr{C}^N$ est la catégorie quotient de $\mathscr{C}$ par les morphismes négligeables par rapport a la trace droite modifiée. Fondamentalement, nous conservons les memes objets, mais augmentons les classes d'isomorphismes des objets, en identifiant des morphismes qui different par un morphisme négligeable. L'effet est que sommer avec un module avec une dimension modifiée nulle ne modifie pas la classe d'isomorphisme. Le point important est que les modules sur le bord de l'alcôve ont la dimension modifiée nulle. Nous montrons qu'au niveau des classes d'isomorphisme des objets simples dans $\mathscr{C}^N$, nous gardons juste les modules $V(n, \tilde{\gamma})$ de $\mathscr{C}$ avec $n \leq l$. Un autre point important concerne la semi-simplicité de la catégorie. Finalement, nous montrons que $\mathscr{C}^N$ est génériquement semi-simple. Le fait que nous ayons évité certains poids $\frac{1}{4}(mod\ \mathbb{Z})$, nous permet de contrôler la décomposition et la semi-simplicité du produit tensoriel pour les petites composantes naturelles des poids. Ensuite, par un argument inductif, nous pouvons contrôler toute la semi-simplicité dans l'alcôve, et une fois que nous atteignons sa limite, nous pouvons ignorer la composante correspondante grâce a la purification que nous avons choisie.

Theorem. *(-, Geer) La catégorie $\mathscr{C}^N$ est une $\mathbb{C}/\mathbb{Z}$-catégorie relative spherique qui conduit a des invariants de Turaev-Viro modifiés pour les 3-variétés.*

III) Algebres centralisatrices liées au $sl(2|1)$ quantique

La troisieme direction de mon doctorat est liée a l'étude des algebres centralisatrices pour des $U_q(sl(2|1))$-représentations. Concernant la représentation de ce super-groupe quantique, les représentations finies et irréductibles de $U_q(sl(2|1))$ avec q générique sont indexées par $\mathbb{N} \times \mathbb{C}$. Comme nous l'avons vu, l'invariant Links-Gould est construit en utilisant la représentation simple en 4 dimensions $V(0,\ alpha)$, correspondant a un parametre complexe générique $\alpha \in \mathbb{C}$. Nous étudions la séquence des algebres centralisatrices correspondant aux puissances tensorielles de la représentation $V(0,\alpha)$.

Soit le poid $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. Avec le $R-$matrix de l'algebre $U_q(sl(2|1))$, on obtient un operateur de Yang Baxter $R \in Aut_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes 2})$. De cette maniere, nous obtenons une séquence de représentations de groupes de tresses:

$$\rho_n : B_n \to Aut_{U_q(sl(2|1))}\left(V(0,\alpha)^{\otimes n}\right) \qquad \rho_n(\sigma_i) = Id^{i-1} \otimes R \otimes Id^{n-i-1}$$

En utilisant la puissance tensorielle de cette représentation, nous obtenons une séquence d'algebres centralisatrices:

$$LG_n(\alpha) := End_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n})$$

En 2011, Marin et Wagner ([67]), prouvent que ce morphisme est surjectif et étudient le noyau pour les petits n. De plus, ils ont montré qu'il factorise a travers une algebre de Hecke cubique notée $H(\alpha)$. Plus loin, ils ont considéré certaines relations qui sont dans le noyau de cette fonction: r_2 pour trois brins et r_3 pour le groupe de tresses a quatre brins. De cette maniere, en utilisant ces relations, ils ont défini un quotient plus petit de l'algebre cubique de Hecke:

$$A_n(\alpha) := H_n(\alpha)/(r_2, r_3)$$

$$\begin{array}{c} \rho_n : \mathbb{C}B_n \to LG_n(\alpha) \\ \searrow \quad \nearrow \\ A_n(\alpha) \end{array}$$

Nous sommes intéressés a étudier les propriétés liées au morphisme ρ_n. Notre motivation pour cela, est sa relation étroite avec l'invariant Links-Gould pour les entrelacs. L'étude de l'algebre $LG_n(\alpha)$ ainsi que la différence entre celle-ci et $\mathbb{C}B_n$ est liée aux relations locales satisfaites par l'opérateur R. En outre, ils ont conjecturé la dimension de l'algebre de centralisateur $LG_n(\alpha)$:

Conjecture 3. *(Conjecture-Marin-Wagner [67])(-)*

$$dim(LG_{n+1}(\alpha)) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

Ils ont également conjecturé que ces relations (r_2 et r_3) sont suffisantes pour générer le noyau de $\rho_n(\alpha)$. En conséquence, on obtiendrait cela:

Conjecture 4. *(Marin-Wagner [67])*

$$A_n(\alpha) \simeq LG_n(\alpha), \forall n \in \mathbb{N}$$

Le résultat principal de cette troisieme partie de la these est la preuve de Conjecture 3. Nous utilisons des techniques combinatoires pour coder la décomposition semi-simple des puissances tensorielles de la représentation canonique 4-dimensional de $U_q(sl(2|1))$. On passe d'abord de la question algébrique initiale du calcul de la dimension de LG_n a un probleme purement combinatoire. Nous construisons certains diagrammes dans le lattice avec des coordonnées entieres sur le plan, ou chaque point a attribué un certain poids. Ceci a pour rôle de coder les dimensions des espaces de multiplicité correspondants associee a $V(0,\alpha)^{\otimes n}$. Inductivement, nous obtenons que les dimensions de ces espaces de multiplicité peuvent etre décrites en utilisant un moyen de compter les chemins dans le plan avec des mouvements prescrits.

Notre stratégie commence par la décomposition semi-simple de $V(0,\alpha)^{\otimes 2}$. De plus, pour les valeurs génériques du parametre $\alpha \in \mathbb{C}$, $V(0,\alpha)^{\otimes n}$ est semi-simple. Ensuite, nous remarquons que tout automorphisme de $V(0,\alpha)^{\otimes n}$ se décomposera en blocs sur les composantes isotypiques correspondant a la décomposition semi-simple de $V(0,\alpha)^{\otimes n}$. Cela montre qu'afin de calculer la dimension de LG_n, il suffit de comprendre cette décomposition semi-simple de la puissance n^{th} tensorielle de $V(0,\alpha)$.

Pour tout $k \in \mathbb{N}$, nous coderons la décomposition de $V(0,\alpha)^{\otimes k}$ en un diagramme $D(k)$ dans le plan. Chaque point $(x,y) \in \mathbb{N} \times \mathbb{N}$ aura un poids $T_k(x,y)$ dans $D(k)$, qui est donné par la multiplicité de $V(x, k\alpha + y)$ dans $V(0,\alpha)^{\otimes k}$. Apres cela, nous codons d'une maniere combinatoire le tenseur avec un $V(0,\alpha)$ supplémentaire, au niveau des diagrammes. La conclusion est que $D(k+1)$ peut etre déterminé a partir de $D(k)$, en effectuant des mouvements locaux a chaque point, ce que nous appelons "blow up". Inductivement, nous obtenons que chaque multiplicité $T_n(x,y)$ est en fait le nombre de chemins dans le plan de longueur $n-1$ avec certains déplacements autorisés. La derniere partie est liée a une correspondance entre ce comptage de chemins, et un autre probleme combinatoire de comptage de paires de chemins dans le plan avec quelques restrictions, pour lesquelles la dimension était connue. Mettre tout cela ensemble conduit a la dimension conjecturée.

Chapter 1

A Homological Model for the coloured Jones polynomials

1.1 Introduction

In this part we will present a topological model for the coloured Jones polynomials. The main tools in our construction are the Lawrence representation $\mathcal{H}_{n,m}$, a dual Lawrence representation $\mathcal{H}^{\partial}_{n,m}$ (which are constructed using the homology of a covering $\tilde{C}_{n,m}$ of the configuration space in the punctured disc $C_{n,m}$) and a geometric intersection pairing between them. Both the Lawrence representation and its dual are generated by homology classes of lifts of m-dimensional Lagrangian submanifolds in the configuration space $\tilde{C}_{n,m}$, called "multiforks" and "barcodes". Furthermore, we study the geometric graded intersection form $<,>$ that exists between the Lawrence representation and its dual and discuss its non-degeneracy for specialisations of coefficients. Using these tools, we construct a sequence of homology classes

$$\tilde{\mathscr{F}}_n^N \in \mathcal{H}_{2n,n(N-1)}|_{\gamma} \quad \text{and} \quad \tilde{\mathscr{G}}_n^N \in \mathcal{H}^{\partial}_{2n,n(N-1)}|_{\gamma}$$

where γ is a certain specialisation of coefficients. Then we show that for any link L, if we consider a braid that represents that link $\beta_{2n} \in B_{2n}$ with $L = \hat{\hat{\beta}}_{2n}$ (oriented plat closure), the N^{th}coloured Jones polynomial is obtained by the geometric pairing in the following manner (1.7.0.1):

$$J_N(L,q) =< \beta_{2n}\mathscr{F}_n^N, \mathscr{G}_n^N > |_{\delta_{N-1}}$$

We would like to mention that γ is a specialisation of the coefficients that does not depend on the colour N, whereas δ_{N-1} is a certain specialisation of the coefficients that depends on N.

Structure of the chapter

In Part 1.2, we present the quantum group $U_q(sl(2))$ that we work with as well as certain properties about its representation theory and the definition of the coloured Jones polynomials. Further on, Section 1.3 contains the details about the homological Lawrence representation. In Section1.4 we present the dual Lawrence representation and we discuss the graded geometric intersection form that relates the two representations, with emphasise on the way of computing this form and the non-degeneracy of this pairing. After that, Part 1.5 concerns identifications between quantum and homological representations of the braid group and contains a detailed discussion about specialisations at natural parameters. Section 1.6, is devoted to the construction and the proof of the homological model for the coloured Jones polynomials. There we construct two homology classes that live in the Lawrence representation specialised by a function that depends on the colour. In the last part, in Section 1.7, we show that the two homology classes can be lifted such that they do not depend on a specialisation using the colour and we conclude the model presented in Theorem 1.7.0.1.

1.2 Representation theory of $U_q(sl(2))$

1.2.1 $U_q(sl(2))$ and its representations

Definition 1.2.1.1. *Let q, s parameters and consider the ring $\mathbb{L}_s := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$.*

Consider the quantum enveloping algebra $U_q(sl(2))$, the algebra over $\mathbb{L}_s$ generated by the elements $\{E, F^{(n)}, K^{\pm 1} |\ n \in \mathbb{N}^*\}$ with the folowing relations:

$$KK^{-1} = K^{-1}K = 1; KE = q^2EK; KF^{(n)} = q^{-2n}F^{(n)}K$$

$$F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}$$

The generators $F^{(n)}$ correspond to the "divided powers" of the generator F, from the version of the quantum group $U_q(sl(2))$ with generators $\{E, F, K^{\pm 1}\}$.

This is a Hopf algebra with the following comultiplication, counit and antipode:

$$\begin{aligned}
\Delta(E) &= E \otimes K + 1 \otimes E, & S(E) &= -EK^{-1} \\
\Delta(F^{(n)}) &= \sum_{j=0}^{n} q^{-j(n-j)} K^{j-n} F^{(j)} \otimes F^{(n-j)}, & S(F^{(n)}) &= (-1)^n q^{n(n-1)} K^n F^{(n)} \\
\Delta(K) &= K \otimes K & S(K) &= K^{-1}, \\
\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} & S(K^{-1}) &= K.
\end{aligned}$$

We will use the following notations:

$$\{x\} := q^x - q^{-x} \qquad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$[n]_q! = [1]_q[2]_q...[n]_q$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

Now we will describe the representation theory of $U_q(sl(2))$. In the sequel the abstract variable s will be thought as being the weight of the Verma module.

Definition 1.2.1.2. *(The Verma module)*

Consider $\hat{V}$ be the $\mathbb{L}_s$-module generated by an infinite family of vectors $\{\hat{v}_0, \hat{v}_1, ...\}$. The following relations define an $U_q(sl(2))$ action on $\hat{V}$:

$$\begin{aligned} K\hat{v}_i =& sq^{-2i}\hat{v}_i, \\ E\hat{v}_i =& \hat{v}_{i-1}, \\ F^{(n)}\hat{v}_i =& \begin{bmatrix} n+i \\ i \end{bmatrix}_q \prod_{k=0}^{n-1}(sq^{-k-i} - s^{-1}q^{k+i})\hat{v}_{i+n}. \end{aligned}$$

1.2.2 Specialisations

For our purpose, to arrive at the definition of the coloured Jones polynomial, it is needed to consider some specialisations of the previous quantum groups and its Verma representations.

Definition 1.2.2.1. *Consider the following specialisations of the coefficients:*
2) Let $h, \lambda \in \mathbb{C}$ and $\boldsymbol{q} = e^h$. In the following, we will have $e^{\lambda h} = \boldsymbol{q}^\lambda$. In this case, we specialise both variable q and the highest weight s to concrete complex numbers:

$$\eta_{\boldsymbol{q},\lambda} : \mathbb{Z}[q^{\pm}, s^{\pm}] \to \mathbb{C}$$
$$\eta_{\boldsymbol{q},\lambda}(q) = e^h \qquad \eta_{\boldsymbol{q},\lambda}(s) = e^{\lambda h}$$

3) This is the case where the coloured Jones polynomial will be defined. Consider q still as a parameter (it will be the parameter from the coloured Jones polynomial), and specialise the highest weight using $\lambda = N - 1 \in \mathbb{N}$ a natural parameter:

$$\eta_\lambda : \mathbb{Z}[q^{\pm}, s^{\pm}] \to \mathbb{Z}[q^{\pm}]$$
$$\eta_\lambda(s) = q^\lambda$$

Using these specialisations, we will consider the corresponding specialised quantum groups and their representation theory. We obtain the following:

Ring	Quantum Group	Representations	Specialisations
$\mathbb{L}_s = \mathbb{Z}[q^{\pm}, s^{\pm}]$	$U_q(sl(2))$	$\hat{V}$	1)q, s param
$\mathbb{C}$	$\mathscr{U}_{\mathrm{q},\lambda} = U_q(sl(2)) \otimes_{\eta_{\mathrm{q},\lambda}} \mathbb{C}$	$\hat{V}_{\mathrm{q},\lambda} = \hat{V} \otimes_{\eta_{q,\lambda}} \mathbb{C}$	2)$(\mathrm{q} = e^h, \lambda) \in \mathbb{C}^2$ $\eta_{\mathrm{q},\lambda}$
$\mathbb{L} = \mathbb{Z}[q^{\pm}]$	$\mathscr{U} = \mathscr{U}_\lambda = U_q(sl(2)) \otimes_{\eta_\lambda} \mathbb{Z}[q^{\pm}]$	$\hat{V}_\lambda = \hat{V} \otimes_{\eta_\lambda} \mathbb{Z}[q^{\pm}]$ $V_N \subseteq \hat{V}_\lambda$	3)q param , $\lambda = N-1 \in \mathbb{N}$ η_λ

Remark 1.2.2.2. *If we specialise as above, $\mathscr{U}_\lambda$ and $\mathscr{U}_{\boldsymbol{q},\lambda}$ become Hopf algebras and $\hat{V}_{\boldsymbol{q},\lambda}$ a $\mathscr{U}_{\boldsymbol{q},\lambda}$-representation and $\hat{V}_\lambda$ a $\mathscr{U}_\lambda$-representation.*

Lemma 1.2.2.3. *If $\lambda = N-1 \in \mathbb{N}$, then $\{\hat{v}_0, ..., \hat{v}_{N-1}\}$ span an N-dimensional $\mathscr{U}_\lambda$-submodule inside $\hat{V}_{N-1}$. Denote this module by*

$$V_N :=< \hat{v}_0, ..., \hat{v}_{N-1} >\subseteq \hat{V}_{N-1}$$

Proof. We can see that K acts by scalars and E decreases the indexes on the basis from before. We only have to see what the generators $F^{(n)}$ do on this space.

$$F^{(n)}\hat{v}_i = \begin{bmatrix} n+i \\ i \end{bmatrix}_q \prod_{k=0}^{n-1} (q^{(N-1)-(k+i)} - q^{-[(N-1)-(k+i)]})\hat{v}_{i+n}.$$

Let $i \in \{0, ..., N-1\}$.

If $\mathbf{n} < \mathbf{N} - \mathbf{i}$, from the definition, the action of $F^{(n)}$ will remain inside the module:

$$F^{(n)}\hat{v}_i \simeq \hat{v}_{i+n} \in V_N.$$

For $\mathbf{n} \geq \mathbf{N} - \mathbf{i}$, we obtain that $n-1 \geq N-1-i$.

This shows us that in the previous formula with the action of $F^{(n)}$, there is a term corresponding to $k = N-1-i$ and its coefficient vanishes.

We obtain that $F^{(n)}\hat{v}_i = 0$, for any $n \geq N-i$.

This concludes the existence of the N-dimensional submodule V_N.

□

1.2.3 The Reshetikhin-Turaev functor

In this section, we will present the general construction due to Reshetikhin and Turaev, that having as input any ribbon category $\mathscr{C}$ gives a functor from the category of tangles to $\mathscr{C}$. In particular, this machinery leads to link invariants. We will present this, using the category of representations of $\mathscr{U}$.

Notation: We will use $U_q(sl(2))\hat{\otimes}U_q(sl(2))$ to denote a completion of the module $U_q(sl(2))\otimes U_q(sl(2))$, where we allow infinite formal sums of tensors.

Proposition 1.2.3.1. *There exist an element $\mathscr{R} \in U_q(sl(2))\hat{\otimes}U_q(sl(2))$ called R-matrix which leads to a braid group representations. For any representation V of $U_q(sl(2))$ (finite dimensional or the Verma module $\hat{V}$), we have that the morphism:*

$$\varphi_n^V : B_n \to \mathrm{End}_{U_q(sl(2))}(V^{\otimes n})$$

such that

$$\sigma_i \longleftrightarrow Id_V^{(i-1)} \otimes (\mathscr{R} \circ \tau) \otimes Id_V^{(n-i-1)}$$
$$\sigma_i^{-1} \longleftrightarrow Id_V^{(i-1)} \otimes (\tau \circ \mathscr{R}^{-1}) \otimes Id_V^{(n-i-1)}$$

gives a well defined action of B_n.
(here $\tau : V\otimes V \to V\otimes V$ flips the two factors between them: $\tau(x\otimes y) = y\otimes x$)

Proposition 1.2.3.2. *1) Using the R-matrix of $U_q(sl(2))$, the category of $U_q(sl(2))$ representations $Rep(\mathscr{U}_q(sl(2)))$ becomes a braided category. More precisely, for any $V, W \in Rep_{\mathscr{U}_q(sl(2))}$, the braiding $\tilde{R}_{V,W} : V \otimes W \to W \otimes V$ is defined using the R-matrix in the following way:*

$$\tilde{R}_{V,W} = (\mathscr{R} \curvearrowright (W \otimes V)) \circ \tau$$

2) The subcategory of finite dimensional $\mathscr{U}$-representations $Rep^{f.\ dim}(\mathscr{U})$ becomes a ribbon category. In this case, the braiding comes from the action of the specialisation of the R-matrix $\mathscr{R}|_{\eta_\lambda\otimes\eta_\lambda} \in \mathscr{U}\hat{\otimes}\mathscr{U}$.

For any representations $V, W \in Rep_{\mathscr{U}}$, the braiding $R_{V,W} : V \otimes W \to W \otimes V$ is defined as:

$$R_{V,W} = (\mathscr{R}|_{\eta_\lambda\otimes\eta_\lambda} \curvearrowright (W \otimes V)) \circ \tau$$

The dualities of this category have the following form:

$$\forall\ V_N \in Rep_{\mathscr{U}}^{f.\ dim}$$

$$\begin{aligned}
\overleftarrow{\mathrm{coev}}_{V_N}&:\mathbb{L}\to V_N\otimes V_N^* \text{ is given by } 1\mapsto \sum v_j\otimes v_j^*,\\
\overleftarrow{\mathrm{ev}}_{V_N}&:V_N^*\otimes V_N\to\mathbb{L} \text{ is given by } f\otimes w\mapsto f(w),\\
\overrightarrow{\mathrm{coev}}_{V_N}&:\mathbb{L}\to V_N^*\otimes V_N \text{ is given by } 1\mapsto \sum v_j^*\otimes K^{-1}v_j, \qquad (1.1)\\
\overrightarrow{\mathrm{ev}}_{V_N}&:V_N\otimes V_N^*\to\mathbb{L} \text{ is given by } v\otimes f\mapsto f(Kv),
\end{aligned}$$

for $\{v_j\}$ a basis of V_N and $\{v_j^*\}$ the dual basis of ${V_N}^*$.

Remark 1.2.3.3. *The action of $\tilde{R}$ on the standard basis of the Verma module $\hat{V}\otimes\hat{V}$ is given in [41](Section 4.1):*

$$\tilde{R}(\hat{v}_i\otimes\hat{v}_j)=s^{-(i+j)}\sum_{n=0}^{i}F_{i,j,n}(q)\prod_{k=0}^{n-1}(sq^{-k-j}-s^{-1}q^{k+j})\ \hat{v}_{j+n}\otimes\hat{v}_{i-n}.$$

In the previous formula $F_{i,j,n}\in\mathbb{Z}[q^{\pm}]$ has the expression:

$$F_{i,j,n}(q)=q^{2(i-n)(j+n)}q^{\frac{n(n-1)}{2}}\begin{bmatrix}n+j\\ j\end{bmatrix}_q.$$

For the Verma module $\hat{V}_{N-1}$, since it is infinite dimensional, we do not have a well defined coevaluation. However, there is the finite dimensional submodule inside it $V_N\subseteq\hat{V}_{N-1}$, which has a coevaluation defined on it. In the sequel, we will define a kind of evaluation on $\hat{V}_{N-1}$, which will be supported on V_N.
Let us make this precise. From the classification of finite dimensional representations of $\mathscr{U}$, it is known that the finite dimensional representations are self-dual.

Lemma 1.2.3.4. *The function $\alpha_N: V_N\to V_N^*$ defined by:*

$$\alpha_N(\hat{v}_i)=(-1)^i q^{-i(N-i)}\hat{v}^*_{N-i-1}$$

is an isomorphism of $\mathscr{U}$-modules.

Notation 1.2.3.5.

$$\textit{Let}\qquad \overrightarrow{\mathrm{Ev}}_{V_N}: V_N^{\otimes 2}\to\mathbb{C} \qquad\qquad \overleftarrow{\mathrm{Coev}}_{V_N}:\mathbb{C}\to V_N^{\otimes 2}$$

$$\overrightarrow{\mathrm{Ev}}_{V_N}:=\overrightarrow{\mathrm{ev}}_{V_N}\circ(Id\otimes\alpha_N) \qquad \overleftarrow{\mathrm{Coev}}_{V_N}:=(Id\otimes\alpha_N^{-1})\circ\overleftarrow{\mathrm{coev}}_{V_N} \qquad (1.2)$$

Definition 1.2.3.6. *Consider* $\overrightarrow{\mathrm{Ev}}_{\hat{V}_{N-1}}\colon \hat{V}_{N-1}^{\otimes 2} \to \mathbb{Z}[q^{\pm}]$ *given by the expression*

$$\overrightarrow{\mathrm{Ev}}_{\hat{V}_{N-1}}\,(\hat{v}_i \otimes \hat{v}_j) = \begin{cases} \overrightarrow{\mathrm{Ev}}_{V_N}\,(\hat{v}_i \otimes \hat{v}_j), \textit{if } \; i,j \leqslant N-1 \\ 0, \qquad\quad \textit{otherwise} \end{cases} \tag{1.3}$$

and extended it by linearity. In other words, this is an extension of the previous evaluation:

$$\overrightarrow{\mathrm{Ev}}_{\hat{V}_{N-1}}\,|_{V_N^{\otimes 2}} = \overrightarrow{\mathrm{Ev}}_{V_N}$$

In the following part we will present the Reshetikhin-Turaev method of obtaining link invariants starting from any ribbon category. Firstly, we will see the definition for the category of coloured tangles.

Definition 1.2.3.7. *Let $\mathscr{C}$ be a category. The category of $\mathscr{C}$-colored framed tangles $T_{\mathscr{C}}$ is defined as follows:*

$$Ob(T_{\mathscr{C}}) = \{(V_1,\epsilon_1),...,(V_m,\epsilon_m) | m \in \mathbb{N}, \epsilon_i \in \pm 1, V_i \in \mathscr{C}\}.$$

$$\mathrm{Hom}_{T_{\mathscr{C}}}\,((V_1,\epsilon_1),...,(V_m,\epsilon_m);(W_1,\delta_1),...,(W_n,\delta_n)) = \mathscr{C}-\textit{colored framed tangles}$$

$$\mathscr{T} : (V_1,\epsilon_1),...,(V_m,\epsilon_m) \uparrow (W_1,\delta_1),...,(W_n,\delta_n)/\textit{isotopy}$$

Remark: The tangles $\mathscr{T}$ have to respect the colors V_i which are at their boundaries. Once we have such a tangle, it has an induced orientation, coming from the signs ϵ_i , using the following conventions:

$$(V,-)\ \downarrow, \quad (V,+)\ \uparrow$$

Theorem 1.2.3.8. *(Reshetikhin-Turaev)*
There exist an unique monoidal functor $\mathbb{F} : T_{Rep^{f.dim}(\mathscr{U})} \to Rep^{f.dim}(\mathscr{U})$ such that $\forall V, W \in Rep^{f.dim}(\mathscr{U})$, it respects the following local relations:

$$1)\ \mathbb{F}((V,+)) = V; \quad \mathbb{F}((V,-)) = V^*$$

$$2)\ \mathbb{F}(\times) = R_{V,W} \in Hom(V \otimes W \to W \otimes V)$$

$$\mathbb{F}(\cup) = \overleftarrow{\mathrm{coev}}_V\colon \mathbb{Z}[q^{\pm}] \to V \otimes V^*$$

$$\mathbb{F}(\cap) = \overrightarrow{\mathrm{ev}}_V\colon V \otimes V^* \to \mathbb{Z}[q^{\pm}]$$

Notation 1.2.3.9. *Let $V \in Rep^{f.dim}(\mathscr{U})$. Consider the subcategory $T_V \subseteq T_{Rep^{f.dim}(\mathscr{U})}$ which contains two objects V and V^* and all tangles are coloured just with these two colours. Denote by*

$$\mathbb{F}_V : T_V \to Rep^{f.dim}(\mathscr{U})$$

the restriction of the Reshetikhin-Turaev functor $\mathbb{F}$ onto this subcategory.

1.2.4 The coloured Jones polynomial $J_N(L,q)$

So far we have seen the algebraic structure coming from $\mathscr{U}$ and the Reshetikhin-Turaev construction. Now, we will present how this machinery is actually a tool that leads to quantum invariants for links.

Definition 1.2.4.1. *(The coloured Jones polynomial-V. Jones)*
Let N be a natural number and L a link. Then $L \in \mathrm{Hom}_{T_{Rep f.dim(\mathscr{U})}}(\emptyset,\emptyset)$.
The N'^{th} coloured Jones polynomial is defined from the Reshetikhin-Turaev functor, using the representation $V_N \in Rep(\mathscr{U})$ as colour, in the following way:

$$J_N(L,q) := \mathbb{F}_{V_N}(L) \in \mathbb{Z}[q^{\pm}]$$

(here by applying the functor we get a morphism from $\mathbb{Z}[q^{\pm}]$ to $\mathbb{Z}[q^{\pm}]$, which is identified with a scalar)

As we have seen so far, the construction that leads to the definition of $J_N(L,q)$ is purely algebraic and combinatorial. We are interested in a geometrical interpretation for this invariant. The method that we are thinking of is to study what is happening with the Reshetikhin-Turaev functor at the intermediary levels of the link diagram. More precisely, we will start with L as a plat closure of a braid $\beta \in B_{2n}$. Then, we will have to study what is happening with $\mathbb{F}$ at three levels:
1) the evaluation $\cap\cap\cap\cap$
2) braid level β
3)the coevaluation $\cup\cup\cup\cup$

The interesting part and the starting point in our description is the fact that at the level of braid group representation, there is a homological counterpart for the quantum representation, called Lawrence representation([60],[57]). This relation is established using the notion of highest weight spaces.

1.2.5 Highest weight spaces

In this part, we will introduce and discuss the properties of some certain vector subspaces which live in the tensor power of a certain representation (we will refer to the ones defined in the table 1.2.7). These subspaces are rich objects and carry very interesting braid group representations, as we will see.

Definition 1.2.5.1. *Consider the set of indexes:*

$$E_{n,m} := \{e = (e_1,...,e_{n-1}) \in \mathbb{N}^{n-1} | e_1 + ... + e_{n-1} = m\}$$

$$E^N_{n,m} := \{e = (e_1, ..., e_{n-1}) \in E_{n,m} | e_1, ..., e_{n-1} \leq N - 1\}$$
$$E^{\geq N}_{n,m} := \{e = (e_1, ..., e_{n-1}) \in E_{n,m} | \exists i,\ e_i \geq N\}$$

For an element $e = (e_1, ..., e_n) \in \mathbb{N}^n$, let us denote:

$$v_e := \hat{v}_{e_1} \otimes ... \otimes \hat{v}_{e_n}$$

Remark 1.2.5.2. *$E_{n,m}$ is the set that has as elements all partitions of the natural number m into $n-1$ natural numbers (possible zero). Its cardinal is well known and we will use the notation:*

$$d_{n,m} := card\ (E_{n,m}) = \binom{n+m-2}{m}$$

Definition 1.2.5.3. *Let $n, m \in \mathbb{N}$ two natural numbers.*
1) The case of two parameters q, s
The weight space of the generic Verma module $\hat{V}$ corresponding to the weight m:

$$\hat{V}_{n,m} := \{v \in \hat{V}^{\otimes n} | Kv = s^n q^{-2m} v\}$$

The highest weight space of the generic Verma module $\hat{V}^{\otimes n}$ corresponding to the weight m:

$$\hat{W}_{n,m} := \hat{V}_{n,m} \cap KerE$$

2) Specialisation with two complex numbers
Let $h, \lambda \in \mathbb{C}$ and $\boldsymbol{q} = e^h$.
The weight space of $\hat{V}^{\otimes n}_{\boldsymbol{q},\lambda}$ corresponding to the weight m:

$$\hat{V}^{\boldsymbol{q},\lambda}_{n,m} := \{v \in \hat{V}^{\otimes n}_{\boldsymbol{q},\lambda} | Kv = q^{n\lambda - 2m} v\}$$

The highest weight space of the Verma module $\hat{V}^{\otimes n}_{\boldsymbol{q},\lambda}$ corresponding to the weight m:

$$\hat{W}^{\boldsymbol{q},\lambda}_{n,m} := \hat{V}^{\boldsymbol{q},\lambda}_{n,m} \cap KerE$$

3)The case with q parameter and λ natural number
a) Inside the Verma module $\hat{V}^{\otimes n}_{N-1}$
Consider $\lambda = N - 1 \in \mathbb{N}$.
The weight space of $\hat{V}^{\otimes n}_{N-1}$ of weight m:

$$\hat{V}^{N-1}_{n,m} := \{v \in \hat{V}^{\otimes n}_{N-1} | Kv = q^{n\lambda - 2m} v\}$$

The highest weight space for Verma module $\hat{V}_{N-1}^{\otimes n}$ corresponding to the weight m:

$$\hat{W}_{n,m}^{N-1} := \hat{V}_{n,m}^{N-1} \cap KerE$$

b) Inside the finite dimensional module $V_N^{\otimes n}$

The weight space for the finite dimensional representation $V_N^{\otimes n}$ of weight m:

$$V_{n,m}^{N} := \{v \in V_N^{\otimes n} | Kv = q^{n(N-1)-2m}v\}$$

The highest weight space of the finite dimensional representation $V_N^{\otimes n}$ corresponding to the weight m:

$$W_{n,m}^{N} := V_{n,m}^{N} \cap KerE$$

Remark 1.2.5.4. *Since $V_N \subseteq \hat{V}_{N-1}$, we have*

$$v_e \in V_N^{\otimes n} \text{ if and only if } e \in E_{n,m}^N.$$

This will happen also at the level of (highest) weight spaces:

$$V_{n,m}^{N} \subseteq \hat{V}_{n,m}^{N-1} \quad \text{and} \quad W_{n,m}^{N} \subseteq \hat{W}_{n,m}^{N-1}.$$

Remark 1.2.5.5. 1) Basis for the weight spaces from Verma module

One can see easily that:

$$\hat{V}_{n,m} =< v_e | e \in E_{n+1,m} > |_{\mathbb{L}_s} \subseteq \hat{V}^{\otimes n}$$

$$\hat{V}_{n,m}^{N-1} =< v_e | e \in E_{n+1,m} > |_{\mathbb{Z}[q^{\pm}]} \subseteq \hat{V}_{N-1}^{\otimes n}$$

Using 1.2.5.2, the dimensions of these space will be:

$$dim(\hat{V}_{n,m}) = dim(\hat{V}_{n,m}^{N-1}) = d_{n+1,m} = \binom{n+m-1}{m}$$

2) Basis for the weight space of the finite dimensional module V_N:

From the previous remark and 1), we conclude that:

$$V_{n,m}^{N} =< v_e | e \in E_{n+1,m}^N > |_{\mathbb{Z}[q^{\pm}]} \subseteq V_N^{\otimes n}$$

Remark 1.2.5.6. *Moreover, if we denote by*

$$V_{n,m}^{\geq N} =< v_e | e \in E_{n+1,m}^{\geq N} >_{\mathbb{L}_s} \subseteq \hat{V}_{N-1}^{\otimes n}$$

$$W_{n,m}^{\geq N} = V_{n,m}^{\geq N} \cap KerE$$

then we have the following splitting as vector spaces:

$$\hat{V}_{n,m}^{N-1} = V_{n,m}^{N} \oplus V_{n,m}^{\geq N}$$

$$\hat{W}_{n,m}^{N} = W_{n,m}^{N} \oplus W_{n,m}^{\geq N}$$

1.2.6 Basis in heighest weight spaces

In [44], there were studied certain bases in the highest weight spaces from the Verma module, as well as connections between highest weight spaces and weight spaces corresponding to different parameters n and m. Jackson and Kerler proved that for the parameter $m = 2$, the braid group action onto the highest weight space $\hat{W}_{n,m}$ corresponds to the homological Lawrence-Bigelow-Krammer ([59], [15],[55],[56])representation and conjectured that this identification is true for any natural number m. Later on, Kohno ([41],[57]) proved this conjecture. We will discuss in details this identification in section 1.5.

Now, we will present from [41] some "good" bases for the highest weight spaces, that will have a role in the identification between quantum and homological representations of the braid groups.

Definition 1.2.6.1. *(Basis for $\hat{W}_{n,m}$) For $e \in E_{n+1,m}$, we will denote by:*

$$v_e^s := s^{\sum_{i=1}^{n} ie_i} \hat{v}_{e_1} \otimes ... \otimes \hat{v}_{e_n}$$

Notice that $\mathscr{B}_{\hat{V}_{n,m}} := \{v_e^s | e \in E_{n+1,m}\}$ form a basis for $\hat{V}_{n,m}$.

In the sequel, the highest weight spaces $\hat{W}_{n,m}$ will be identified with a certain subspace of the weight spaces $\hat{V}_{n,m}$.

Let $\iota : E_{n,m} \to E_{n+1,m}$ the inclusion:

$$\iota((e_1, ..., e_{n-1})) = (0, e_1, ..., e_{n-1})$$

Denote by $\hat{V}'_{n,m} := \mathbb{L}_s \hat{v}_0 \oplus \hat{V}_{n-1,m} \subseteq \hat{V}_{n,m}$. Then, $\mathscr{B}_{\hat{V}'_{n,m}} := \{\hat{v}^s_{\iota(e)} | e \in E_{n,m}\}$ will give a basis for the space $\hat{V}'_{n,m}$.

Proposition 1.2.6.2. *[41] Consider the function $\phi : \hat{V}'_{n,m} \to \hat{W}_{n,m}$ described by the formula:*

$$\phi(w) := \sum_{k=0}^{m} (-1)^k s^{-k(n-1)} q^{2mk-k(k+1)} v_k \otimes E^k(w)$$

Then ϕ is an isomorphism of $\mathbb{L}_s$-modules.

The set $\mathscr{B}_{\hat{W}_{n,m}} = \{\phi(v^s_{\iota(e)}) | e \in E_{n,m}\}$ will describe a basis for the generic highest weight space $\hat{W}_{n,m}$. It follows that (1.2.5.2,1.2.5.5):

$$dim\ (\hat{W}_{n,m}) = d_{n,m} = \binom{n+m-2}{m}.$$

1.2.7 Quantum representations of the braid groups

In the following part, we will see that the braid group action on (generic) Verma module and on the finite dimensional module $V_N^{\otimes n}$, passes at the level of highest weight spaces.

Proposition 1.2.7.1. *Since $\varphi_n^{\hat{V}}$ gives an action on $\hat{V}^{\otimes n}$ which commutes with the quantum group action (Prop. 1.2.3.1), it will commute with the actions of generators K, E. Then, it will induce a well defined action on the generic highest weight spaces $\varphi_{n,m}^{\hat{W}} : B_n \to \mathrm{Aut}(\hat{W}_{n,m})$.*

This action in the basis $\mathscr{B}_{\hat{W}_{n,m}}$ will lead to a representation:

$$\varphi_{n,m}^{\hat{W}} : B_n \to GL(d_{n,m}, \mathbb{L}_s)$$

This is called the generic quantum representation on highest weight spaces of the Verma module.

Proposition 1.2.7.2. *Similarly, using the previous specialisations we have induced braid group actions :*

$$2) \quad \varphi_{n,m}^{\hat{W}^{\boldsymbol{q},\lambda}} : B_n \to \mathrm{Aut}(\hat{W}_{n,m}^{\boldsymbol{q},\boldsymbol{\lambda}})$$

well defined action induced by $\varphi_n^{\hat{V}_{\boldsymbol{q},\lambda}}$

$$3)a) \ \varphi_{n,m}^{\hat{W}^{N-1}} : B_n \to \mathrm{Aut}(\hat{W}_{n,m}^{N-1})$$

well defined action induced by $\varphi_n^{\hat{V}_{N-1}}$ called the quantum representation on highest weight spaces of the Verma module.

$$3)b) \ \varphi_{n,m}^{W^N} : B_n \to \mathrm{Aut}(W_{n,m}^N)$$

well defined action induced by $\varphi_n^{V_N}$ called the quantum representation on highest weight spaces of the finite dimensional module.

As a summary we have the following highest weights spaces, which carry braid group actions and live inside the n^{th} tensor power of different specialisations of the Verma module $\hat{V}$:

Braid group action	Highest weight space	Representation	Specialisation
$\varphi^{\hat{W}}_{n,m}$	$\hat{W}_{n,m}$	$\hat{V}^{\otimes n}$	1) q, s param
$\varphi^{\hat{W}^{\mathrm{q},\lambda}}_{n,m}$	$\hat{W}^{\mathrm{q},\lambda}_{n,m}$	$\hat{V}^{\otimes n}_{\mathrm{q},\lambda}$	2)$\mathrm{q} = e^h, \lambda \in \mathbb{C}$ $\eta_{\mathrm{q},\lambda}$
$\varphi^{\hat{W}^{N-1}}_{n,m}$	$\hat{W}^{N-1}_{n,m}$	$\hat{V}^{\otimes n}_{N-1}$	3)a) q param $\lambda = N-1 \in \mathbb{N}$ η_λ
$\varphi^{W^N}_{n,m}$	$W^N_{n,m}$	$V^{\otimes n}_N$	3)b) q param $\lambda = N-1 \in \mathbb{N}$ η_λ

1.3 Lawrence representation

1.3.1 Local system

In this section we will present certain braid group representations introduced by Lawrence ([60]). These are defined on the middle homology of a certain covering of the configuration space in the punctured disk. They are called homological Lawrence representations and they have a topological description.

Let $n \in \mathbb{N}$. Consider $\mathcal{D}^2 \subseteq \mathbb{C}$ the unit disk with its boundary and $\{p_1, ..., p_n\}$- n points in its interior, on the real axis.

Let $D_n := \mathcal{D}^2 \setminus \{p_1, ..., p_n\}$ and fix $m \in \mathbb{N}$ a natural number.
Let $C_{n,m}$ be the unordered configuration space of m points in the n-punctured disk:

$$C_{n,m} = Conf_m(\mathcal{D}_n) = (\mathcal{D}_n^m \setminus \{x = (x_1, ..., x_n) \mid \exists\, i, j \text{ such that } x_i = x_j\})/Sym_m$$

(by Sym_m we denote the symmetric group of order m)

Definition 1.3.1.1. *(Local system on $C_{n,m}$)*
Consider the abelianisation

$$ab : \pi_1(C_{n,m}) \to H_1(C_{n,m}).$$

Then $H_1(C_{n,m}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}$, where each i-th element of $\mathbb{Z}^n$ is generated by a loop around the puncture p_i and the last component counts the total winding number of the loop with respect to itself, viewed in the configuration space $C_m(\mathcal{D}^2)$.
Consider the function

$$aug : \mathbb{Z}^n \oplus \mathbb{Z} \to \underset{< x >}{\mathbb{Z}} \oplus \underset{< d >}{\mathbb{Z}}$$

by taking the sum on the first components: $aug((x_1, ..., x_n), y) = (x_1 + ... + x_n, y)$.

By composing the previous maps, define the local system :

$$\phi : \pi_1(C_{n,m}) \to \mathbb{Z} \oplus \mathbb{Z}$$

$$\phi = aug \circ ab$$

We denote $\tilde{C}_{n,m}$ be the covering of $C_{n,m}$corresponding to $Ker(\phi)$ and its associated projection map $\pi : \tilde{C}_{n,m} \to C_{n,m}$.

Remark 1.3.1.2. *The deck transformations of the covering are:*

$$Deck(\tilde{C}_{n,m}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Then each deck transformation will induce a cellular chain map for $\tilde{C}_{n,m}$ and moreover this map will pass at the level of homology. So we have an action

$$\mathbb{Z} \oplus \mathbb{Z} \curvearrowright H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z}).$$

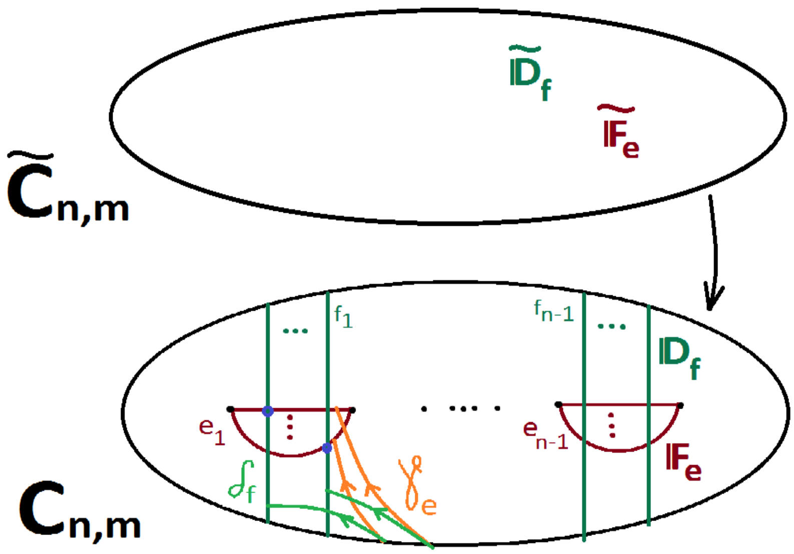
Moreover, this action will be defined at the level of the group ring:

$$\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \simeq \mathbb{Z}[x^{\pm}, d^{\pm}]$$

It follows that the homology groups $H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ and $H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$ have a structure of $\mathbb{Z}[x^{\pm}, d^{\pm}]$-modules (here H^{lf} means the Borel-Moore homology/ the homology of locally finite chains).

1.3.2 Basis of multiforks

So far, we have seen the covering of the configuration space whose homology will lead to the Lawrence representation. For that, we will define certain subspaces in the homology (Borel-Moore/relative to the boundary) ([22]) of $\tilde{C}_{n,m}$.



Definition 1.3.2.1. *(***Multiforks***)[18],[41]*

1) Submanifolds

Let $e=(e_1,...,e_{n-1})\in E_{n,m}$ *as in the definition 1.2.5.1. To each e it will be associate an m-dimensional submanifold in* $\tilde{C}_{n,m}$ *which will give a class in the Borel Moore homology.*

Fix $d_1,...,d_m\in\partial D_n$*.*

For each $i\in\{1,...,n-1\}$*, consider* e_i*-disjoint horizontal segments in* D_n*, between* p_i *and* p_{i+1} *(which meet just at their boundary). Denote those segments by* $I^e_1,...,I^e_{e_1},...,I^e_m$*. Also, for each* $k\in\{1,...,m\}$*, choose a vertical path* γ^e_k *between the segment* I^e_k *and* d_k*.*

Think each segment as a map $I^e_i:(0,1)\rightarrow D_n$*. Since these segments are disjoint, their product gives a map:*

$$I^e_1\times...\times I^e_m:(0,1)^m\rightarrow D^m_n\setminus\{x=(x_1,...,x_n)|x_i=x_j\})$$

Let the projection defined by the quotient with respect to the Sym_m*-action:*

$$\pi_m:D^m_n\setminus\{x=(x_1,...,x_n)|x_i=x_j\})\rightarrow C_{n,m}$$

Composing the two previous maps we obtain: $\mathbb{F}_e : \mathcal{D}^m(=(0,1)^m) \to C_{n,m}$

$$\mathbb{F}_e = \pi_m \circ (I_1^e \times ... \times I_m^e)$$

2) Base Points*: The paths to the base points* $d_1, .., d_m$ *will help us to lift the submanifold* $\mathbb{F}_e$ *in* $\tilde{C}_{n,m}$*. The* m*-uple* $(d_1, ..., d_m)$ *defines a point* $\mathbf{d} \in C_{n,m}$*. Consider* $\tilde{\mathbf{d}} \in \pi^{-1}(\mathbf{d})$*. The product of the paths towards the boundary* γ_k^e*, will define a path in the configuration space. Let*

$$\gamma^e := \pi_m \circ (\gamma_1^e, ..., \gamma_m^e) : [0,1] \to C_{n,m}$$

Consider the unique lift of the path γ^e *such that* $\tilde{\gamma}^e(0) = \tilde{\mathbf{d}}$*:*

$$\tilde{\gamma}^e : [0,1]^m \to \tilde{C}_{n,m}$$

3) Multiforks *Consider* $\tilde{\mathbb{F}}_e$ *to be the unique lift of* $\mathbb{F}_e$ *to the covering which passes through the point* $\tilde{\gamma}^e(1)$*:*

$$\tilde{\mathbb{F}}_e : \mathcal{D}^m(=(0,1)^m) \to \tilde{C}_{n,m}$$

Then $\tilde{\mathbb{F}}_e$ *will define a class in the Borel-Moore Homology* $[\tilde{\mathbb{F}}_e] \in H_m^{lf}(\tilde{C}_{n,m}, \mathbb{Z})$*.*

$[\tilde{\mathbb{F}}_e]$ *is called the multifork corresponding to the element* $e \in E_{n,m}$

The Lawrence representation will be a subspace of this Borel-Moore homology of the covering, spanned by these multiforks. More precisely we have the following definition:

Definition 1.3.2.2. *Consider the subspace:*

$$\mathcal{H}_{n,m} := < [\tilde{\mathbb{F}}_e] \mid e \in E_{n,m} >_{\mathbb{Z}[x^{\pm}, d^{\pm}]} \subseteq H_m^{lf}(\tilde{C}_{n,m}, \mathbb{Z}) \qquad 1.3.1.2$$

Denote by $\mathscr{B}_{\mathcal{H}_{n,m}} := \{[\tilde{\mathbb{F}}_e] \mid e \in E_{n,m}\}$*.*

Proposition 1.3.2.3. *From ([41],Prop 3.1)* $\mathcal{H}_{n,m}$ *is a free module over* $\mathbb{Z}[x^{\pm}, d^{\pm}]$ *of dimension* $d_{n,m}$ *and* $\mathscr{B}_{\mathcal{H}_{n,m}}$ *describes a basis called multifork basis.*

As we have seen, the cardinal of $E_{n,m}$ is known (1.2.5.2), so we have:

$$\text{rank}(\mathcal{H}_{n,m}) = d_{n,m} = \binom{n+m-2}{m}.$$

1.3.3 Braid group action

Now we are interested about the relation between this subspace which lives in the homology of the covering of the configuration space and the braid group action on the punctured disc. It is known ([49], chap. I.6) that:

$$B_n = MCG(D_n) = Homeo^+(D_n, \partial)/\text{isotopy}$$

Then $B_n \curvearrowright C_{n,m}$ and it will induce an action

$$B_n \curvearrowright \pi_1(C_{n,m}).$$

Remark 1.3.3.1. *Let $\pi : E \to B$ be a covering map corresponding to a local system $\phi : \pi_1(B, x_0) \to H$ where H is an abelian group.*

Suppose that G is a group that acts on B.

We will consider the fiber of E over a point $x \in B$ in the following way:

$$\pi^{-1}(x) = \{\textit{classes of paths from the fixed point } x_0 \textit{ to } x\} \diagup \simeq$$

where $\sigma_1 \simeq \sigma_2$ iff $\phi(\sigma_1 \sigma_2^{-1}) = 0$.

Then this action can be lifted to a G-action on E (constructed at the level of paths using the previous definition) if and only if

$$\forall g \in G, \quad g(ker(\phi)) \subseteq ker(\phi)$$

From the definition of the local system 1.3.1.1 it can be shown that $\forall \beta \in B_n$:

$$\beta(ker(\phi)) \subseteq ker(\phi)$$

.

Remark 1.3.3.2. *1) It follows that there is a well defined action of the braid group B_n on the covering of the configuration space $\tilde{C}_{n,m}$.*

2)We are interested to study the homology of this covering(1.3.1.2). One can check that the action $B_n \curvearrowright \tilde{C}_{n,m}$ commutes with the action of the Deck transformations $< x, d >$.

Moreover, it can be shown that ϕ is the finest abelian local system such that the induced action of the braid group on the corresponding covering 1.3.3.1 commutes with the deck transformations given by H.

For us, the base space B will be the configuration space $C_{n,m}$ and the group $H = \mathbb{Z} \oplus \mathbb{Z}$, as in 1.3.1.1.

Corollary 1.3.3.3. *From the previous remarks, one can conclude that there is a well defined action:*

$$B_n \curvearrowright H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z}) \text{ (} as\ a\ \mathbb{Z}[x^{\pm}, d^{\pm}] - module).$$

Definition 1.3.3.4. *([41] Prop 3.1) (Lawrence representation)*

The subspace $\mathcal{H}_{n,m} \subseteq H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ is invariant under the action of B_n.

Considering the braid group action on $\mathcal{H}_{n,m}$ in the multifork basis $\mathscr{B}_{\mathcal{H}_{n,m}}$, it is obtained a representation which is called the Lawrence representation:

$$l_{n,m} : B_n \to GL(d_{n,m}, \mathbb{Z}[x^{\pm}, d^{\pm}]) \ \ (= End(\mathcal{H}_{n,m}, \mathbb{Z}[x^{\pm}, d^{\pm}])).$$

1.4 Blanchfield pairing

In this section, we will present a non-degenerate duality between the Lawrence representation $\mathcal{H}_{n,m}$ and a "dual" space, which we will denote by $\mathcal{H}^{\partial}_{n,m}$. This dual space lives in the homology of the covering relative to its boundary. Using this form, we will be able to express any element in the dual of $\mathcal{H}_{n,m}$, as certain geometric pairing, using elements from the dual space. This property will play an important role in the homological model from Section 1.6.

1.4.1 Dual space

Firstly we will define a subset in the homology of the covering of the configuration space relative to its boundary $H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$, by specifying a generating set, which we will think as a dual set to the multifork basis.

Definition 1.4.1.1. *(***Barcodes***)[18]*

1) Submanifolds *Let $e = (e_1, ..., e_{n-1}) \in E_{n,m}$ (1.2.5.1). For each such e, we will define an m-dimensional submanifold in $\tilde{C}_{n,m}$, which will give a homology class in $H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$.*

For each $i \in \{1, ..., n-1\}$, consider e_i-disjoint vertical segments in D_n, between p_i and p_{i+1} as in the picture above. Denote those segments by $J^e_1, ..., J^e_{e_1}, ..., J^e_m$. Also, for each $k \in \{1, ..., m\}$, choose a vertical path δ^e_k between the segment J^e_k and d_k.

Each of these segments is a map $J^e_i : [0,1] \to D_n$. Then the product of these segments leads to a map:

$$J^e_1 \times ... \times J^e_m : [0,1]^m \to D^m_n \setminus \{x = (x_1, ..., x_n) | x_i = x_j\}.$$

Projecting onto the configuration space using π_n *we obtain a submanifold:*

$$\mathbb{D}_e : (\bar{\mathcal{D}}^m(= [0,1]^m), \partial\bar{\mathcal{D}}^m) \to (C_{n,m}, \partial C_{n,m}).$$

2) Base Points*: As in the case of multiforks, the paths to the base point* $\mathbf{d} \in C_{n,m}$ *will help us to lift the submanifold* $\mathbb{D}_e$ *to the covering* $\tilde{C}_{n,m}$*. Consider the path in the configuration space:*

$$\delta^e := \pi_m \circ (\delta_1^e, ..., \gamma_m^e) : [0,1] \to C_{n,m}.$$

Define $\tilde{\delta}^e$ *to be the unique lift of the path* δ^e *such that* $\tilde{\delta}^e(0) = \tilde{\mathbf{d}}$*:*

$$\tilde{\delta}^e : [0,1] \to \tilde{C}_{n,m}.$$

.

3) Barcodes *Consider* $\tilde{\mathbb{D}}_e$ *to be the unique lift of* $\mathbb{D}_e$ *to the covering which passes through* $\tilde{\delta}^e(1)$*:*

$$\tilde{\mathbb{D}}_e : \mathcal{D}^m \to \tilde{C}_{n,m}.$$

Then $\tilde{\mathbb{D}}_e$ *will define a class in the homology relative to the boundary* $[\tilde{\mathbb{D}}_e] \in H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$*.*

$[\tilde{\mathbb{D}}_e]$ *is called the barcode corresponding to the element* $e \in E_{n,m}$*.*

Definition 1.4.1.2. *(The "dual" representation)*
Let the subspace generated by all the barcodes:

$$\mathcal{H}^{\partial}_{n,m} :=< [\tilde{\mathbb{D}}_e] \mid e \in E_{n,m} >_{\mathbb{Z}[x^{\pm}, d^{\pm}]} \subseteq H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial) \qquad 1.3.1.2.$$

We will call this the "dual" representation of $\mathcal{H}_{n,m}$*. Also, consider the set:*

$$\mathscr{B}_{\mathcal{H}^{\partial}_{n,m}} := \{[\tilde{\mathbb{D}}_e] | e \in E_{n,m}\}.$$

Remark 1.4.1.3. *We do not know yet that* $\mathscr{B}_{\mathcal{H}^{\partial}_{n,m}}$ *is a basis for* $\mathcal{H}^{\partial}_{n,m}$*, but we will prove this in the next section, using a pairing between* $\mathcal{H}_{n,m}$ *and* $\mathcal{H}^{\partial}_{n,m}$*.*

1.4.2 Graded Intersection Pairing

In this part, we will describe how the Borel-Moore homology and the homology relative to the boundary of $\tilde{C}_{n,m}$ are related by a pairing. More precisely we are interested to define a Blanchfield type pairing between $\mathcal{H}_{n,m}$ and $\mathcal{H}^{\partial}_{n,m}$.

We will present a duality type pairing, which uses the middle dimensional homologies with respect to differents parts of the "boundary" of the covering space. We will use the space $\tilde{C}_{n,m}$ and think about its boundary as having two parts. The first part, the "boundary at infinity", contains the multi-points in $\tilde{C}_{n,m}$, where one of their components projects to D_n "close to a puncture" or where two components get very close one to another by projection. The second part, is the actual boundary and contains the multi-points for which there exists a component which projects onto the boundary of D_n.

The Borel-Moore homology of $\tilde{C}_{n,m}$, can be thought as the homology with respect to the first boundary from above, relative to infinity. The second homology that we will use will be the homology with respect to the boundary of $\tilde{C}_{n,m}$, as described above as the second set.

We will follow [17], [18], especially the way of computing the pairing in the case when the homology classes are given by some manifolds. Let us take two homology classes $[\tilde{M}] \in H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ and $[\tilde{N}] \in H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$ which can be represented by the classes of lifts of two m-dimensional submanifolds $M, N \subseteq C_{n,m}$. The idea is to fix the second submanifold $\tilde{N}$ in the covering and act with all deck transformations on the first submanifold $\tilde{M}$. Each time, we will count the geometric intersection between the two submanifolds with the coefficient given by the element from the deck group. Recall that the local system is defined as $\phi : \pi_1(C_{n,m}) \to \mathbb{Z} \oplus \mathbb{Z}$ and $Deck(\tilde{C}_{n,m}) = \mathbb{Z} \oplus \mathbb{Z}$.

Definition 1.4.2.1. *[16](Graded intersection)*
Let $F \in H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ and $G \in H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$. Suppose that there exist $M, N \subseteq C_{n,m}$ transverse submanifolds of dimension m which intersect in a finite number of points such that there exist lifts in the covering $\tilde{M}, \tilde{N}$ with

$$F = [\tilde{M}] \quad \textit{and} \quad G = [\tilde{N}].$$

Then the graded intersection between the submanifolds $\tilde{M}$ and $\tilde{N}$ is defined by the formula:

$$<< \tilde{M}, \tilde{N} >>:= \sum_{(u,v) \in \mathbb{Z} \oplus \mathbb{Z}} (x^u d^v \curvearrowright \tilde{M} \cap \tilde{N}) \cdot x^u d^v \in \mathbb{Z}[x^{\pm}, d^{\pm}]$$

where $(\cdot \cap \cdot)$ means the geometric intersection number between submanifolds.

Remark 1.4.2.2. *For any $\varphi \in Deck(\tilde{C}_{n,m})$:*

$$\varphi \tilde{M} \cap \tilde{N} \subseteq \pi^{-1}(M \cap N)$$

This shows that the previous sum has a finite number of non-zero terms and the graded geometric intersection between $\tilde{M}$ and $\tilde{N}$ is well defined.

In the sequel, we will see that the graded intersection between $\tilde{M}$ and $\tilde{N}$ which is a priori defined in the covering $\tilde{C}_{n,m}$, can be computed in the base, using M and N and the local system for coefficients. More specifically, the pairing will be described as a sum parametrised by all intersection points of M and N in $C_{n,m}$, where for each point it will be counted a coefficient which is prescribed by the local system.

Proposition 1.4.2.3. *Let $x \in M \cap N$. Then there exists an unique $\varphi_x \in Deck(\tilde{C}_{n,m})$ such that*

$$(\varphi_x \tilde{M} \cap \tilde{N}) \cap \pi^{-1}(x) \neq \varnothing.$$

Proof. The fact that N is a submanifold guarantees that $\forall y \in N$:

$$\text{card } |\tilde{N} \cap \pi^{-1}(\{y\})| = 1.$$

Let us denote $\tilde{y}_{\tilde{N}} := \tilde{N} \cap \pi^{-1}(\{y\})$ and $\tilde{y}_{\tilde{M}} := \tilde{M} \cap \pi^{-1}(\{y\})$ (we use the same property for M as well). Then it follows that:

$$(\varphi \tilde{M} \cap \tilde{N}) \cap \pi^{-1}(\{x\}) \neq 0 \text{ iff } \tilde{x}_{\tilde{N}} \in \varphi \tilde{M}$$

From this, we conclude that if φ satisfies the required condition, then

$$\varphi(\tilde{x}_{\tilde{M}}) = \tilde{x}_{\tilde{N}}$$

From the properties of the Deck transformation, this is a characterisation for an unique φ_x. □

The last two remarks show that the intersection points between all the translations of $\tilde{M}$ by the deck transformations and $\tilde{N}$ are actually parametrised by the intersection points between M and N:

$$\bigcup_{\varphi \in Deck(\tilde{C}_{n,m})} (\varphi \tilde{M} \cap \tilde{N}) \longleftrightarrow M \cap N$$

Computation[16] We will fix a basepoint $d \in C_{n,m}$ and $\tilde{d} \in \pi^{-1}(d)$. From the last part, we notice that in order to compute the pairing $<< \tilde{M}, \tilde{N} >>$, it is enough to consider a sum parametrised by the set $M \cap N$ and see which is the corresponding coefficient for each intersection point. Let $x \in M \cap N$ and $\varphi_x \in Deck(\tilde{C}_{n,m})$ as in 1.4.2.3. Denote by $\tilde{x} = (\varphi\tilde{M} \cap \tilde{N}) \cap \pi^{-1}(x)$. Now we will describe φ_x using just the local system and the point x. We notice that we have the same sign of the intersection in the covering and in the base:

$$(\varphi_x\tilde{M} \cap \tilde{N})_{\tilde{x}} = (M \cap N)_x$$

Denote this sign by c_x. Consider two paths $\gamma_M : [0,1] \to C_{n,m}$ and $\delta_N : [0,1] \to C_{n,m}$ such that if we take the unique lifts of which start in $\tilde{d}$ of those paths
$\tilde{\gamma}_M, \tilde{\delta}_N : [0,1] \to \tilde{C}_{n,m}$ we have the following properties:

$$\gamma_M(0) = d; \quad \gamma_M(1) \in M; \quad \tilde{\gamma}_M(1) \in \tilde{M}$$

$$\delta_N(0) = d; \quad \delta_N(1) \in N; \quad \tilde{\delta}_N(1) \in \tilde{N}$$

After that, let us denote by $\hat{\gamma}_M, \hat{\delta}_N : [0,1] \to C_{n,m}$ such that

$$Im(\hat{\gamma}_M) \subseteq M; \quad \hat{\gamma}_M(0) = \gamma_M(1); \quad \hat{\gamma}_M(1) = x$$

$$Im(\hat{\delta}_N) \subseteq N; \quad \hat{\delta}_N(0) = \delta_N(1); \quad \hat{\delta}_N(1) = x$$

We can consider the loop:

$$l_x := \delta_N \hat{\delta}_N \hat{\gamma}_M^{-1} \gamma_M^{-1}$$

Proposition 1.4.2.4. *[16]*

$$\varphi_x = \phi(l_x)$$

Corollary 1.4.2.5. *The pairing between $\tilde{M}$ and $\tilde{N}$ can be computed using just the submanifolds in the base space $C_{n,m}$ and the local system:*

$$<< \tilde{M}, \tilde{N} >>= \sum_{x \in M \cap N} c_x \phi(l_x) \in \mathbb{Z}[x^{\pm}, d^{\pm}]$$

This pairing $<<, >>$ can be defined in a similar way for homology classes $F \in H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ and $G \in H_m(\tilde{C}_{n,m}, \mathbb{Z}; \partial)$ that can be represented as linear combinations of homology classes of lifts of submanifolds of the type that we described above, with the condition of a finite set of intersection points.

Lemma 1.4.2.6. *([18](6.2)) The paring $<< F, G >>$ does not depend on the representants of the homology classes, so it is well defined at the level of homology.*

1.4.3 Pairing between $\mathcal{H}_{n,m}$ and $\mathcal{H}^{\partial}_{n,m}$

Definition 1.4.3.1. *Let us consider the Blanchfield pairing:*

$$<,>: \mathcal{H}_{n,m} \otimes \mathcal{H}^{\partial}_{n,m} \rightarrow \mathbb{Z}[x^{\pm}, d^{\pm}]$$

$$< [\tilde{\mathbb{F}}_e], [\tilde{\mathbb{D}}_f] >=<< \tilde{\mathbb{F}}_e, \tilde{\mathbb{D}}_f >>$$

which will define a sesquilinear form (with respect to the transformations $x \leftrightarrow x^{\pm}, d \leftrightarrow d^{-1}$).

Lemma 1.4.3.2. *For any $e, f \in E_{n,m}$, the pairing has the following form:*

$$< [\tilde{\mathbb{F}}_e], [\tilde{\mathbb{D}}_f] >= p_e \cdot \delta_{e,f}$$

where $p_e \in \mathbb{N}[d^{\pm}]$ and $p_e \neq 0$ with a non-zero constant term.

Proof. We remark that working in the configuration space $\mathbb{F}_e \cap \mathbb{D}_f = \emptyset$ if $e \neq f$. Let $e \in E_{n,m}$.

$$<< \tilde{\mathbb{F}}_e, \tilde{\mathbb{D}}_e >>= \sum_{x \in \mathbb{F}_e \cap \mathbb{D}_e} c_x \phi(l_x) \in \mathbb{Z}[x^{\pm}, d^{\pm}]$$

Secondly, we notice that the previous intersection can be computed using the separate intersections between the submanifolds from $\mathbb{F}_e$ and $\mathbb{D}_e$ "supported" between punctures i and $i+1$, in the following manner:

$$<< \tilde{\mathbb{F}}_e, \tilde{\mathbb{D}}_e >>= \prod_{i=1}^{n-1} << \tilde{\mathbb{F}}_{e_i}, \tilde{\mathbb{D}}_{e_i} >>$$

where $\mathbb{F}_{e_i} := \mathbb{F}_{(0,0,...,e_i,...,0)}$ and $\mathbb{D}_{e_i} := \mathbb{D}_{(0,0,...,e_i,...,0)}$.

Now we will compute $<< \tilde{\mathbb{F}}_{e_i}, \tilde{\mathbb{D}}_{e_i} >>$. We notice that each intersection point $x \in \mathbb{F}_{e_i} \cap \mathbb{D}_{e_i}$ is characterised by an e_i-uple which pairs a horizontal line from the multifork with a vertical line from the barcode. In other words, x is determined by a permutation on the grid $\sigma_x \in S_{e_i}$. It follows:

$$<< \tilde{\mathbb{F}}_{e_i}, \tilde{\mathbb{D}}_{e_i} >>= \sum_{\sigma \in S_{e_i}} c_\sigma \phi(l_\sigma)$$

The geometric intersection sign c_σ counts whether $\mathbb{F}_{e_i}$ and $\mathbb{D}_{e_i}$ have a positive or a negative intersection in $x = \big(x_{(1,\sigma(1))}, ..., x_{(e_i,\sigma(e_i))}\big)$. The configuration space on the disc is orientable. Let us consider $\mathscr{R} = \{v^1, v^2\}$ the standard base for the tangent space of the disc. Let $c = (c_1, ..., c_m) \in C_{n,m}$ and a tangent vector in this point w. We will define the orientation of w by writing it into the form $(w^1_{c_1},...,w^m_{c_m},w^2_{c_1},...,w^2_{c_m})$ and see if written in the canonical base $\mathscr{R}$ has the same sign or not as the vector $(v^1_{c_1},....,v^1_{c_m},v^2_{c_1},...,v^2_{c_m})$. This is well defined at the level of configuration space, because we are working on a manifold of even dimension, so if we change the order of points by a transposition, we will have to modify the matrix with an even number negative signs.

Following this recipe, we see that c_σ is the sign of the tangent vector v_σ obtained by taking the tangent vectors at the multiforks followed by the tangent vectors at the barcode:

$$v_\sigma = (v^1_{x_{(1,\sigma(1))}},....,v^1_{x_{(e_i,\sigma(e_i))}},v^2_{x_{(1,\sigma(1))}},...,v^2_{x_{(e_i,\sigma(e_i))}})$$

Here, we used that all segments of the multifork are oriented in the same way, and also, that all parts of the barcode have the same orientation. We conclude that $c_\sigma = 1$.

Now we will look at the polynomial part from the graded intersection. Following the previous description of computation, for any $k \in 1,...,m$ let:

$$\hat{\gamma}^e_k \subseteq I_k \text{ such that } \hat{\gamma}^e_k(0) = \gamma^e_k(1);\ \hat{\gamma}^e_k(1) = x_{(k,\sigma(k))}$$

$$\hat{\delta}^e_k \subseteq J_k \text{ such that } \hat{\delta}^e_k(0) = \delta^e_k(1);\ \hat{\delta}^e_k(1) = x_{(k,\sigma(k))}$$

Let us denote $a_i := e_1 + ... + e_{i-1}$ and the following paths in the configuration space:

$$\Gamma_{e_i} := \left(\gamma^e_{a_i+1},...,\gamma^e_{a_i+e_i}\right) \qquad \hat{\Gamma}_{e_i} := \left(\hat{\gamma}^e_{a_i+1},...,\hat{\gamma}^e_{a_i+e_i}\right)$$

$$\Delta_{e_i} := \left(\delta^e_{a_i+1},...,\delta^e_{a_i+e_i}\right) \qquad \hat{\Delta}_{e_i} := \left(\hat{\delta}^e_{a_i+1},...,\hat{\delta}^e_{a_i+e_i}\right)$$

Then, the loop corresponding to σ has the following form:

$$l_\sigma = \Delta_{e_i}\hat{\Delta}_{e_i}\hat{\Gamma}^{-1}_{e_i}\Gamma^{-1}_{e_i}$$

Firstly we see that l_σ does not goes around any of the punctures, so the variable x from the local system will not appear. Secondly, for $\sigma = Id$ the path l_{Id} is the union of trivial loops and so $\phi(l_{Id}) = 1$.
From the formula 1.4.3 and the previous remarks, we conclude that:

$$<< \tilde{\mathbb{F}}_{e_i}, \tilde{\mathbb{D}}_{e_i} >> \in \mathbb{N}[d^{\pm}]$$

and has a nontrivial free term. Combining this with the computation 1.4.3, we conclude that

$$< [\tilde{\mathbb{F}}_e], [\tilde{\mathbb{D}}_e] > \in \mathbb{N}[d^{\pm}]$$

with a non trivial free part, which concludes the proof. $\square$

This shows that all the polynomials p_e are not zero divisors in $\mathbb{Z}[x^{\pm}, d^{\pm}]$.

Definition 1.4.3.3. *Let $e, f \in E_{n,m}$ and consider $[\tilde{\mathbb{F}}_e] \in \mathcal{H}_{n,m}$, $[\tilde{\mathbb{D}}_f] \in \mathcal{H}^{\partial}_{n,m}$*

Lemma 1.4.3.4. *The family of barcodes $\{[\tilde{\mathbb{D}}_e] | e \in E_{n,m}\}$ is linearly independent and it forms a basis for $\mathcal{H}^{\partial}_{n,m}$.*

Proof. Let $\alpha_1, ..., \alpha_{d_{n,m}} \in \mathbb{Z}[x^{\pm}, d^{\pm}]$ and suppose that $\sum_{i=1}^{d_{n,m}} \alpha_i [\tilde{\mathbb{D}}_{e_i}] = 0 \in \mathcal{H}^{\partial}_{n,m}$.

Consider $j \in \{1, ..., d_{n,m}\}$. After we make the pairing with the multifork $[\tilde{\mathbb{F}}_{e_j}]$ we obtain:

$$< [\tilde{\mathbb{F}}_{e_j}], \sum_{i=1}^{d_{n,m}} \alpha_i [\tilde{\mathbb{D}}_{e_i}] > = < [\tilde{\mathbb{F}}_{e_j}], \alpha_{e_j} [\tilde{\mathbb{D}}_{e_j}] > = \alpha_j \cdot p_{e_j} = 0$$

Since the polynomials p are not zero divisors in $\mathbb{Z}[x^{\pm}, d^{\pm}]$, we conclude that $\alpha_j = 0, \forall j \in \{1, ..., d_{n,m}\}$. $\square$

Notation 1.4.3.5. *The set $\mathscr{B}_{\mathcal{H}^{\partial}_{n,m}}$ will be called the barcodes basis for $\mathcal{H}^{\partial}_{n,m}$.*

Remark 1.4.3.6. *By an analog argument, we re-obtain also a proof for the fact that the multiforks $\{[\tilde{\mathbb{F}}_e] | e \in E_{n,m}]$ are linearly independent in $H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$. (see[43]-3.1)*

Remark 1.4.3.7. *From the previous computation, we get the matrix of the graded intersection pairing $<,>$ in the bases of multiforks $\mathscr{B}_{\mathcal{H}_{n,m}}$ and barcodes*

$\mathscr{B}_{\mathcal{H}^{\partial}_{n,m}}$:

$$M_{<,>} = \begin{bmatrix} p_{e_1} & 0 & \dots & & 0 \\ 0 & p_{e_2} & 0 & \dots & 0 \\ & & & & \\ & & & & \\ 0 & \dots & & 0 & p_{e_{d_{n,m}}} \end{bmatrix}$$

(where $p_1, ..., p_{e_{d_{n,m}}} \in \mathbb{Z}[x^{\pm}, d^{\pm}]$ are all non-zero divisors).

Corollary 1.4.3.8. *The Blanchfield pairing is a non-degenerate sesquilinear form:*

$$<,>: \mathcal{H}_{n,m} \otimes \mathcal{H}^{\partial}_{n,m} \to \mathbb{Z}[x^{\pm}, d^{\pm}]$$

1.4.4 Specialisations

Our aim is to describe the coloured Jones polynomials in a homological way. For this purpose, our starting point is the deep connection that relates the quantum representations of the braid groups and certain specializations of the Lawrence representations. In this part we will focus in order to define and study Blanchfield pairings which are defined on those specializations of the Lawrence representation that are used in Kohno's Theorem.

Definition 1.4.4.1. *Let $\lambda = N - 1 \in \mathbb{N}$ a parameter.*
1)Consider the specialization of the coefficients $\psi_{\boldsymbol{q},\lambda}$ defined by:

$$\psi_{\lambda} : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}]$$

$$\psi_{\lambda}(x) = q^{2\lambda}, \quad \psi_{\lambda}(d) = -q^{-2}$$

2) The specialized Lawrence representation and its dual will have the following definition:

$$\mathcal{H}_{n,m}|_{\psi_\lambda} = \mathcal{H}_{n,m} \otimes_{\psi_\lambda} \mathbb{Z}[q^{\pm}] =< [\tilde{\mathbb{F}}_e] | e \in E_{n,m}] >_{\mathbb{Z}[q^{\pm}]}$$

and these multiforks will define a basis of $\mathcal{H}_{n,m}|_{\psi_\lambda}$ over $\mathbb{Z}[q^{\pm}]$, using 1.4.3.6

$$\mathcal{H}^{\partial}_{n,m}|_{\psi_\lambda} = \mathcal{H}^{\partial}_{n,m} \otimes_{\psi_\lambda} \mathbb{Z}[q^{\pm}] =< [\tilde{\mathcal{D}}_e] | e \in E_{n,m}] >_{\mathbb{Z}[q^{\pm}]}$$

and these barcodes will define a basis of $\mathcal{H}_{n,m}|_{\psi_\lambda}$ over $\mathbb{Z}[q^{\pm}]$, using 1.4.3.4

Definition 1.4.4.2. *Let us consider a specialised Blanchfield pairing, obtained from the generic pairing $<,>$ by specialising its coefficients using ψ_λ.*

$$<,> |_{\psi_\lambda} : \mathcal{H}_{n,m}|_{\psi_\lambda} \otimes \mathcal{H}^{\partial}_{n,m}|_{\psi_\lambda} \to \mathbb{Z}[q^{\pm}]$$

$$< [\tilde{\mathbb{F}}_e], [\tilde{\mathbb{D}}_f] > |_{\psi_\lambda} = \psi_\lambda(p_e) \cdot \delta_{e,f}$$

Remark 1.4.4.3. *We notice that $\{p_e | e \in E_{n,m}\} \cap Ker(\psi_\lambda) = \varnothing$.*

1)Here we see that the choice of barcodes on the dual side of $\mathcal{H}_{n,m}$ has an important role. They lead to the non-zero polynomials $p \in \mathbb{N}[d^{\pm}]$ on the diagonal of the matrix $M_{<,>}$ of the geometric intersection pairing $<,>$. This fact, ensures that these polynomials become non-zero elements in $\mathbb{Z}[q^{\pm}]$ through the specialization ψ_λ.

2) It would be interesting to compare this situation with the case where we use dual-noodles (noodles with multiplicities) instead of barcodes. In that case, the generic pairing will have as coefficients on the diagonal, polynomials $p \in \mathbb{Z}[x^{\pm}, d^{\pm}]$, which are really in 2 variables and moreover have $\mathbb{Z}$ coefficients not only $\mathbb{N}$ coefficients. In our geometric model for the coloured Jones polynomial $J_N(L, \boldsymbol{q})$, we will use the specialisation ψ_{N-1} with natural parameter $\lambda = N-1 \in \mathbb{N}$. In this case, some of these diagonal polynomials might become zero through the specialisation ψ_{N-1} because this change of coefficients essentially impose the relation $x = -d^{-\lambda}$ and $\lambda = N-1 \in \mathbb{N}$.

3)An interesting question is to understand the pairing in the noodle case and to compute its kernel.

Corollary 1.4.4.4. *The form $<,> |_{\psi_\lambda}$ is sesquilinear and non-degenerate over $\mathbb{Z}[q^{\pm}]$.*

1.4.5 Dualizing the algebraic evaluation

This part is motivated by the fact that we are interested to describe the third level of a plat closure of a braid (the union of "caps") viewed through the Reshetikhin-Turaev functor, in a geometrical way using the geometric intersection pairing. We will see the details of this in the folowing section 1.6, but for this part the aim is to be able to understand an element of the dual of $\mathcal{H}_{n,m}|_{\psi_\lambda}$, as a geometric intersection $< \cdot, \mathscr{G} >$ for some $G \in \mathcal{H}_{n,m}|_{\psi_\lambda}$.

Remark 1.4.5.1. *The pairing* $<,>|_{\psi_\lambda} : \mathcal{H}_{n,m}|_{\psi_\lambda} \otimes \mathcal{H}^{\partial}_{n,m}|_{\psi_\lambda} \to \mathbb{Z}[q^{\pm}]$ *is non-degenerate and has the matrix:*

$$M_{<,>} = \begin{bmatrix} \psi_\lambda(p_{e_1}) & 0 & \dots & & 0 \\ 0 & \psi_\lambda(p_{e_2}) & 0 & \dots & 0 \\ & & & & \\ & & & & \\ 0 & \dots & & 0 & \psi_\lambda(p_{e_{d_{n,m}}}) \end{bmatrix}$$

(where $\psi_\lambda(p_1), \dots, \psi_\lambda(p_{e_{d_{n,m}}}) \in \mathbb{Z}[q^{\pm 2}]$ are polynomials with non-zero free term).

In particular, this shows that the diagonal coefficients of the pairing are not necessary invertible elements in $\mathbb{Z}[q^{\pm}]$.

Problem 1.4.5.2. *From this, we see that a priori, not any element of $\mathcal{F} \in (\mathcal{H}_{n,m}|_{\psi_\lambda})^*$ can be described as a geometric intersection pairing $< \cdot, \mathscr{G}_{\mathcal{F}} >$ for some $G_{\mathcal{F}} \in \mathcal{H}^{\partial}_{n,m}|_{\psi_\lambda}$. This issue comes from the fact that we are working over a ring and not over a field. In order to overcome this problem, in the sequel we will change the coefficients ring $\mathbb{Z}[q^{\pm}]$ by passing to the field of fractions $\mathbb{Q}(q)$.*

We remember the specialisation $\psi_\lambda : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}]$ described by:

$$\psi_\lambda(x) = q^{2\lambda} \quad \psi_\lambda(d) = -q^{-2}$$

Let us consider the embedding $i : \mathbb{Z}[q^{\pm}] \hookrightarrow \mathbb{Q}(q)$ and use $\mathbb{Q}(q)$ as field of coefficients.

Definition 1.4.5.3. *(New Specialisation)*
1)Let the specialization $\alpha_\lambda : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$ defined by:

$$\alpha_\lambda = i \circ \psi_\lambda$$

2)Let the specialised Lawrence representations defined in a similar way as before:

$$\mathcal{H}_{n,m}|_{\alpha_\lambda} := \mathcal{H}_{n,m} \otimes_{\alpha_\lambda} \mathbb{Q}(q) =< [\tilde{\mathbb{F}}_e] | e \in E_{n,m}] >_{\mathbb{Q}(q)}$$

and the multiforks define a basis of $\mathcal{H}_{n,m}|_{\alpha_\lambda}$ over $\mathbb{Q}(q)$, from 1.4.3.6

$$\mathcal{H}^{\partial}_{n,m}|_{\alpha_\lambda} = \mathcal{H}^{\partial}_{n,m} \otimes_{\alpha_\lambda} \mathbb{Q}(q) =< [\tilde{\mathbb{F}}_e] | e \in E_{n,m}] >_{\mathbb{Q}(q)}$$

and the barcodes define a basis of $\mathcal{H}_{n,m}|_{\alpha_\lambda}$ over $\mathbb{Q}(q)$, from 1.4.3.4

We notice that, in fact the specialisations are related in the following manner:

$$\mathcal{H}_{n,m}|_{\alpha_\lambda} := \mathcal{H}_{n,m}|_{\psi_\lambda} \otimes_i \mathbb{Q}(q); \qquad \mathcal{H}^{\partial}_{n,m}|_{\alpha_\lambda} := \mathcal{H}^{\partial}_{n,m}|_{\psi_\lambda} \otimes_i \mathbb{Q}(q)$$

Also, let us denote $p_\lambda : \mathcal{H}_{n,m}|_{\psi_\lambda} \to^{(\ \cdot\ \otimes_i 1)} \mathcal{H}_{n,m}|_{\alpha_\lambda}$ the corresponding change of the coefficients.

Definition 1.4.5.4. *Consider in a similar way as before a specialised Blanchfield pairing, by specialising the pairing $<,>$ using α_λ:*

$$<,> |_{\alpha_\lambda} : \mathcal{H}_{n,m}|_{\alpha_\lambda} \otimes \mathcal{H}^{\partial}_{n,m}|_{\alpha_\lambda} \to \mathbb{Q}(q)$$

$$< [\tilde{\mathbb{F}}_e], [\tilde{\mathbb{D}}_f] > |_{\alpha_\lambda} = \alpha_\lambda(p_e) \cdot \delta_{e,f}$$

Remark 1.4.5.5. *For any $e \in E_{n,m}$, $\alpha_\lambda(p_e) \in \mathbb{Q}(q)$ is a non-zero element, so it is invertible. This shows that $<,> |_{\alpha_{q,\lambda}}$ is a non-degenerate sesquilinear form.*

Moreover, working on a field, we conclude that any element in the dual of the first space, will be described as a pairing with a fixed element from the second space.

Corollary 1.4.5.6. *For any* $G \in (\mathcal{H}_{n,m}|_{\alpha_\lambda})^*$*, there exist a homology class* $\mathscr{G} \in \mathcal{H}^{\partial}_{n,m}|_{\alpha_\lambda}$ *such that:*

$$G =< \cdot, \mathscr{G} > |_{\alpha_\lambda}$$

Remark 1.4.5.7. *(Construction of geometric duals)*
Start with an element $G_0 \in (\mathcal{H}_{n,m}|_{\psi_\lambda})^* = \mathrm{Hom}(\mathcal{H}_{n,m}|_{\psi_\lambda}, Z[q^{\pm}])$*. Construct the following corresponding element:*

$$G := G_0 \otimes Id_{\mathbb{Q}(q)} \in (\mathcal{H}_{n,m}|_{\alpha_\lambda})^*.$$

If we consider the pairing with the dual element $\mathscr{G}$ *of* G *given by 1.4.5.6, we obtain* G_0 *in a geometrical way:*

$$G_0 \otimes Id_{\mathbb{Q}(q)} =< \cdot, \mathscr{G} > |_{\alpha_\lambda}$$

1.5 Identifications between quantum representations and homological representations

So far, we have presented two important constructions that lead to representations of the braid group: the quantum representation and the Lawrence representation. A priori, they are defined using totally different tools, the quantum representation comes from the algebraic world whereas the Lawrence representation has a homological description. In this section we will discuss about those, using a result due to Kohno that relates these two representations.

Let $h, \lambda \in \mathbb{C}$ and $\mathrm{q} = e^h$. Let us consider the folowing two specialisations of the coefficients using these complex numbers:

1) for the quantum representation $\hat{W}$ (defined over $\mathbb{Z}[q^{\pm}, s^{\pm}]$):

$$\eta_{\mathrm{q},\lambda} : \mathbb{Z}[q^{\pm 1}, s^{\pm 1}] \to \mathbb{C}$$

$$\eta_{\mathrm{q},\lambda}(q) = \mathrm{q}; \quad \eta_{\mathrm{q},\lambda}(s) = \mathrm{q}^{\lambda}$$

2) for the Lawrence representation $\mathcal{H}_{n,m}$ (defined over $\mathbb{Z}[x^{\pm}, d^{\pm}]$):

$$\psi_{\mathrm{q},\lambda} : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{C}$$
$$\psi_{\mathrm{q},\lambda}(x) = \mathrm{q}^{2\lambda}; \quad \psi_{\mathrm{q},\lambda}(d) = -\mathrm{q}^{-2}.$$

Kohno relates these two representations, by connecting each of them with a monodromy representation of the braid group which arises using the theory of KZ-connections. We will shortly describe these relations, following [41].

1.5.1 KZ-Monodromy representation

Let the Lie algebra $sl_2(\mathbb{C})$ and consider an orthonormal basis $\{I_\mu\}_\mu$ for its Cartan-Killing form. Denote by

$$\Omega = \sum_{\mu} I_\mu \otimes I_\mu \in sl(2) \otimes sl(2).$$

Notation 1.5.1.1. *1) For $\lambda \in \mathbb{C}^*$ consider M_λ to be the Verma module of $sl(2)$,*
$M_\lambda = < v_0, v_1, ... >_{\mathbb{C}}$ with the following actions:

$$Hv_i = (\lambda - 2i)v_i$$
$$Ev_i = v_{i-1}$$
$$Fv_i = (i+1)(\lambda - i)v_{i+1}.$$

2)Denote by $X_n = \mathbb{C}^n \setminus \left(\bigcup_{1 \leq i,j \leq n} Ker(z_i = z_j)\right)$ and $Y_n := X_n / S_n$.
3) Let $n \in \mathbb{N}$ and consider the endomorphism

$$\Omega_{i,j} \in End(M_\lambda^{\otimes n})$$

to be the action of Ω onto the i^{th} and j^{th} components.

Definition 1.5.1.2. *(KZ-connection) Let $h \in \mathbb{C}^*$ be a parameter.*
Consider ω_h the following 1-form defined over Y_n with values into $End(M_\lambda^{\otimes n})$, called the KZ-connection (Knizhnik-Zamolodchikov):

$$\omega_h = \frac{h}{\sqrt{-1}\ \pi} \sum_{1 \leq i,j \leq n} \Omega_{i,j} \frac{dz_i - dz_j}{z_i - z_j}$$

This describes a connection which is flat with values into the trivial bundle $Y_n \times M_\lambda^{\otimes n}$ over Y_n.

After that, the monodromy of this connection will lead to a representation:

$$\nu_h : B_n \to Aut(M_\lambda^{\otimes n})$$

Definition 1.5.1.3. *(Space of null vectors)*
Let $m \in \mathbb{N}$. The space of null vectors in $M_\lambda^{\otimes n}$ corresponding to the weight m is described using the sl_2-action in the following way:

$$N[n\lambda - 2m] := \{v \in M_\lambda^{\otimes n} | Ev = 0; Hv = (n\lambda - 2m)v\}$$

Definition 1.5.1.4. *(Monodromy representation from ω_h and M_λ)*
For any $m \in \mathbb{N}$, the monodromy of the KZ-connection ω_h will induce a braid group representation on the spaces of null vectors:

$$\nu_h : B_n \to Aut(N[n\lambda - 2m]).$$

Proposition 1.5.1.5. *[41] For $e \in E_{n,m}$, consider the vector*

$$w_e := \sum_{i=0}^{m} (-1)^i \frac{1}{\lambda(\lambda-1)\cdot \ldots \cdot (\lambda - i)} F^i v_0 \otimes E^i (F^{e_1} v_0 \otimes \ldots \otimes F^{e_{n-1}} v_0).$$

Then, for any $\lambda \in \mathbb{C}^* \setminus \mathbb{N}$, the following set describes a basis of $N[n\lambda - 2m]$:

$$\mathscr{B}_{N[n\lambda-2m]} := \{w_e | e \in E_{n,m}\}$$

Remark 1.5.1.6. *For natural parameter $\lambda \in \mathbb{N}$, $\mathscr{B}_{N[n\lambda-2m]}$ is not even well defined.*

Theorem 1.5.1.7. *[41],[57](Kohno's Theorem)*
There exist an open dense set $U \subseteq \mathbb{C}^* \times \mathbb{C}^*$ such that for any $(h, \lambda) \in U$ there is the following identification of representations of the braid group:

$$\left(\hat{W}_{n,m}^{q,\lambda}, \mathscr{B}_{\hat{W}_{n,m}^{q,\lambda}}\right) \simeq_{\Theta_{q,\lambda}} \left(\mathcal{H}_{n,m}|_{\psi_{q,\lambda}}, \mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_{q,\lambda}}\right)$$

More precisely, the quantum representation $\varphi_{n,m}^{\hat{W}^{q,\lambda}}$ and Lawrence representation $l_{n,m}|_{\psi_{q,\lambda}}$ are the same in the bases described above (1.2.7.1,1.3.1.2).

1.5.2 Identifications with q and λ complex numbers

We are interested in understanding the quantum representations with natural parameter $\lambda = N-1 \in \mathbb{N}$. In this case, we are not anymore in the "generic parameters" case. For that, we will study the relation between the previous braid group representations specialised with any parameters.

We will start with some general remarks about the group actions on modules and how they behave with respect to specialisations.

Remark 1.5.2.1. *Let R be a ring and M an R-module with a fixed basis $\mathscr{B}$ of cardinal d. Consider a group action $G \curvearrowright M$ and a representation of G using the basis $\mathscr{B}$:*

$$\rho : G \to GL(d, R).$$

Suppose that S is another ring and we have a specialisation of the coefficients, given by a ring morphism:

$$\psi : R \to S$$

Denote: $M^{\psi} := M \otimes_R S$ and $\mathscr{B}_{M^{\psi}} := \mathscr{B} \otimes_R 1 \in M^{\psi}$. From this, we will have an induced group action $G \curvearrowright M^{\psi}$.
Then, we have the following properties:
1) $\mathscr{B}_{M^{\psi}}$ is a basis for M^{ψ}.
2)Let $\rho^{\psi} : G \to GL(d, S)$ the representation of G on M^{ψ} coming from the induced action, in the basis $\mathscr{B}_{M^{\psi}}$. In this way, the two actions, before and after specialisation give the same action in the following sense:

$$\rho^{\psi}(g) = \rho(g)|_{\psi} \quad \forall g \in G$$

(here if $f : M \to M$, denote by $f|_{\psi} : M^{\psi} \to M^{\psi}$ the specialisation $f|_{\psi} = f \otimes_R Id_S$)

Comment 1.5.2.2. *We are interested into the case of non-generic complex parameters $(h, \lambda) \in \mathbb{C}^* \times \mathbb{C}$. We would like to to emphasise that quantum representation and Lawrence representation on one side and the KZ-monodromy representation on the other have different natures with respect to the complex parameters (h, λ). Actually, both quantum representation $\varphi_{n,m}^{\hat{W}^{\boldsymbol{q},\lambda}}$ and Lawrence representation $l_{n,m}|_{\psi_{\boldsymbol{q},\lambda}}$ are coming from some generic braid group representations $\varphi_{n,m}^{\hat{W}}$ and $l_{n,m}$ and then are specialised using the procedure from the previous remark for the functions $\eta_{\boldsymbol{q},\lambda}$ and $\psi_{\boldsymbol{q},\lambda}$. On the contrary, in order to obtain the KZ-monodromy representation η_h, one has to fix the complex* *numbers (λ, h) and do all the construction through this parameters. This is not globalised in a way that does not depend on the specific values, in the sense that we can't construct a representation over some abstract variables such that the KZ-representation at the complex parameters can be obtained from the abstract one by a specialisation, as in the previous remark.*

Problem 1.5.2.3. *Since the KZ representation does not comes from a specialisation procedure, we see that we do not have a well defined action in a well defined basis for any complex parameters. From the remark 1.5.1.6, for $\lambda \in \mathbb{N}$ a natural parameter, $\mathscr{B}_{N[n\lambda-2m]}$ is not even a well defined set in $N[n\lambda - 2m]$. However, the isomorphism between the quantum and homological representations still works for any parameters, using a continuity argument.*

Theorem 1.5.2.4. *Let $(h, \lambda) \in \mathbb{C}^* \times \mathbb{C}$ fixed parameters. Then the following braid group representations are isomorphic, using the following corresponding bases:*

$$\left(\hat{W}^{q,\lambda}_{n,m}, \mathscr{B}_{\hat{W}^{q,\lambda}_{n,m}}\right) \simeq_{\Theta_{q,\lambda}} \left(\mathcal{H}_{n,m}|_{\psi_{q,\lambda}}, \mathscr{B}_{\mathcal{H}_{n,m}|_{\psi_{q,\lambda}}}\right)$$

Proof. 1) In the proof of 1.5.1.7, there are glued two identifications between representations of the braid group B_n. Basically, the relation between the quantum representation and the Lawrence representation is established by passing from both of them to the monodromy of the KZ-connection. There are constructed two isomorphisms of braid group representations:

$$f^{WN}_{\mathbf{q},\lambda} : \mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}} \to N[n\lambda - 2m]$$

$$f^{NH}_{\mathbf{q},\lambda} : N[n\lambda - 2m] \to \hat{W}^{\mathbf{q},\lambda}_{n,m}$$

More precisely, those isomorphisms are proved using correspondences between the following bases:

$$\begin{array}{ccccc}
\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m} & & \nu_h & & l_{n,m}|_{\psi_{\mathbf{q},\lambda}} \\[1em]
\hat{W}^{\mathbf{q},\lambda}_{n,m} & \simeq & N[n\lambda - 2m] & \simeq & \mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}} \\
\mathscr{B}_{\hat{W}^{\mathbf{q},\lambda}_{n,m}} & & \mathscr{B}_{N[n\lambda-2m]} & & \mathscr{B}_{\mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}}} \\
\phi(v^s_e)|_{\eta_{\mathbf{q},\lambda}} & \leftarrow & w_e & \leftarrow & [\tilde{\mathbb{F}}_e] \\
& f^{NH}_{\mathbf{q},\lambda} & & f^{WN}_{\mathbf{q},\lambda} &
\end{array}$$

2)From this, Kohno proved that for any pair of parameters $(h,\lambda)\in U$:

$$\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m}(\beta)=l_{n,m}|_{\psi_{\mathbf{q},\lambda}}(\beta),\qquad \forall \beta\in B_n$$

3) Let us denote by

$$\Theta_{\mathbf{q},\lambda}:\mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}}\to \hat{W}^{\mathbf{q},\lambda}_{n,m}$$

$$\Theta_{\mathbf{q},\lambda}([\tilde{\mathbb{F}}_e])=\phi(v^s_e)|_{\eta_{\mathbf{q},\lambda}},\forall e\in E_{n,m}$$

This function is defined for all $(h,\lambda)\in\mathbb{C}\times\mathbb{C}$. We notice that, having in mind that they are defined directly on the bases, the functions $f^{WN}_{\mathbf{q},\lambda}, f^{NH}_{\mathbf{q},\lambda}$ are continuous with respect to the parameters $(h,\lambda)\in U$. This means that the function $\Theta_{\mathbf{q},\lambda}$ is continuous with respect to the two complex parameters.

Now, we will see what is happening with non-generic parameters.

4) We are interested to see what is happening to the specialisation of the quantum representation.

We know that $\mathscr{B}_{\hat{W}_{n,m}}$ is a basis for $\hat{W}_{n,m}$. Making the specialisation $\eta_{\mathbf{q},\lambda}$, means to take a tensor product, which will ensure that $\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_{\mathbf{q},\lambda}}$ will still describe a basis for the specialised module. We conclude that

$$\mathscr{B}_{\hat{W}^{\mathbf{q},\lambda}_{n,m}}:=\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_{\mathbf{q},\lambda}}$$

is a well defined basis of $\hat{W}^{\mathbf{q},\lambda}_{n,m}$, for any $(h,\lambda)\in\mathbb{C}^*\times\mathbb{C}$.

5) Since the specialisation $\eta_{\mathbf{q},\lambda}$ is well defined for any complex parameters $(h,\lambda)\in\mathbb{C}^*\times\mathbb{C}$, all the coefficients from $\varphi^{\hat{W}}_{n,m}|_{\eta_{\mathbf{q},\lambda}}$ will become well defined complex numbers. In particular $\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m}$ in the basis $\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_{\mathbf{q},\lambda}}$ has all the coefficients well defined.

6) Using the previous steps 4) and 5), we conclude that for any braid $\beta\in B_n$, the specialisation of the matrix obtained from the initial action $\varphi^{\hat{W}}_{n,m}$ onto $\hat{W}_{n,m}$ in the basis $\mathscr{B}_{\hat{W}_{n,m}}$, is actually the matrix of the specialised action $\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m}$ in the specialised basis $\mathscr{B}_{\hat{W}^{\mathbf{q},\lambda}_{n,m}}$:

$$\varphi^{\hat{W}}_{n,m}(\beta)|_{\eta_{\mathbf{q},\lambda}}=\underset{\mathscr{B}_{\hat{W}^{\mathbf{q},\lambda}_{n,m}}}{\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m}(\beta)},\qquad \forall (h,\lambda)\in(\mathbb{C}^*\times\mathbb{C})$$

7) The set $\mathscr{B}_{\mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}}}$ is well defined and describes a basis for $\mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}}$ for any parameters $(h,\lambda)\in\mathbb{C}^*\times\mathbb{C}$ (1.4.3.6).

8)This shows that for every $\beta \in B_n$, the specialisations of the matrices from the action on $\mathcal{H}_{n,m}$ in the multifork basis, are actually the same as the matrices of the specialised Lawrence action, in the specialised multifork basis $\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_{\mathbf{q},\lambda}}$:

$$l_{n,m}(\beta)|_{\psi_{\mathbf{q},\lambda}} = \underset{\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_{\mathbf{q},\lambda}}}{l_{n,m}|_{\psi_{\mathbf{q},\lambda}}(\beta)}, \qquad \forall(\mathbf{q},\lambda) \in (\mathbb{C}^* \times \mathbb{C})$$

Combining the points 2), 3), 6), 8) we obtain that for any parameters $(q,\lambda) \in \mathbb{C}^* \times \mathbb{C}$, we have the identification:

$$\varphi_{n,m}^{\hat{W}^{\mathbf{q},\lambda}}(\beta) = l_{n,m}|_{\psi_{\mathbf{q},\lambda}}(\beta), \quad \forall \beta \in B_n$$

This concludes that the quantum representation and the Lawrence representation are isomorphic for any parameters. $\square$

1.5.3 Identifications with q indeterminate

From the previous discussion, we know that the quantum representation $\hat{W}_{n,m}$ and the Lawrence representation $l_{n,m}$ are isomorphic after appropriate identifications of the coefficients, as long as we fix (q, λ) complex numbers. In the sequel, we will state a similar result, but for the case where we keep q as an indeterminate.

Definition 1.5.3.1. *Let us fix $\lambda = N-1 \in N$ and q an indeterminate. Consider the specialisations of the coefficients:*

$$\eta_\lambda : \mathbb{Z}[q^{\pm 1}, s^{\pm 1}] \to \mathbb{Z}[q^{\pm}]$$
$$\eta_\lambda(s) = q^\lambda$$
$$\psi_\lambda : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}]$$
$$\psi_\lambda(x) = q^{2\lambda}; \quad \psi_\lambda(d) = -q^{-2}$$

For $\boldsymbol{q} \in \mathbb{C}$, let $f_{\boldsymbol{q}} : \mathbb{Z}[q^{\pm}] \to \mathbb{C}$

$$f_{\boldsymbol{q}}(q) = \boldsymbol{q}$$

Then we notice that the specialisations are related:

$$\eta_{\boldsymbol{q},\lambda} = f_{\boldsymbol{q}} \circ \eta_\lambda$$

$$\psi_{\boldsymbol{q},\lambda} = f_{\boldsymbol{q}} \circ \psi_\lambda$$

We remind the notations:

$$\hat{W}^{\lambda}_{n,m} = \hat{W}_{n,m} \otimes_{\eta_\lambda} \mathbb{Z}[q^{\pm}]$$

$$\mathcal{H}_{n,m}|_{\psi_\lambda} = \mathcal{H}_{n,m} \otimes_{\psi_\lambda} \mathbb{Z}[q^{\pm}]$$

Theorem 1.5.3.2. *The braid group representations over $\mathbb{Z}[q^{\pm}]$ are isomorphic:*

$$\left(\hat{W}^{\lambda}_{n,m}, \mathscr{B}_{\hat{W}^{\lambda}_{n,m}}\right) \simeq_{\Theta_\lambda} \left(\mathcal{H}_{n,m}|_{\psi_\lambda}, \mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_\lambda}\right)$$

Proof. We will basically use the Theorem 1.5.2.4, and just study a little more its proprieties. Let

$$\Theta_\lambda : \mathcal{H}_{n,m}|_{\psi_\lambda} \to \hat{W}_{n,m}|_{\eta_\lambda}$$

$$\Theta_\lambda(\tilde{\mathbb{F}}_e) = \phi(v^s_e)|\eta_\lambda, \forall e \in E_{n,m}$$

1) We notice that $\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_\lambda}$ is well defined and, as in the proof of Theorem 1.5.2.4, it will define a basis in $\hat{W}^{\lambda}_{n,m}$.
2)Similarly, $\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_\lambda}$ is a basis of $\mathcal{H}_{n,m}|_{\psi_\lambda}$.
3) Actually we have the relations:

$$\hat{W}^{\mathbf{q},\lambda}_{n,m} = \hat{W}^{\lambda}_{n,m} \otimes_{f_{\mathbf{q}}} \mathbb{C}$$

$$\mathcal{H}_{n,m}|_{\psi_{\mathbf{q},\lambda}} = \mathcal{H}_{n,m}|_{\psi_\lambda} \otimes_{f_{\mathbf{q}}} \mathbb{C}$$

4) If we take $\beta \in B_n$, we notice that for any $\mathbf{q} \in \mathbb{C}$:

$$\varphi^{\hat{W}^{\mathbf{q},\lambda}}_{n,m}(\beta) = f_{\mathbf{q}}\left(\varphi^{\hat{W}}_{n,m}(\beta)|_{\eta_\lambda}\right)$$

$$l_{n,m}(\beta)|_{\psi_{\mathbf{q},\lambda}} = f_{\mathbf{q}}\left(l_{n,m}(\beta)|_{\psi_\lambda}\right)$$

(here, the sense is that $f_{\mathbf{q}} : M(d_{n,m}, \mathbb{Z}[q^{\pm}]) \to M(d_{n,m}, \mathbb{C})$, by specialising every entry of the matrix using the function $f_{\mathbf{q}}$).
5)This shows that

$$\varphi^{\hat{W}}_{n,m}(\beta)|_{\eta_\lambda} = l_{n,m}(\beta)|_{\psi_\lambda}, \forall \beta \in B_n$$

6)Up to this point, this is just an equality of matrices. The question now, is whether these are matrices of the actions $\varphi^{\hat{W}}_{n,m}|_{\eta_\lambda}$ and $l_{n,m}|_{\psi_\lambda}$ in some well defined basis.

Puting everything together, we conclude that the vector spaces $\hat{W}^{\lambda}_{n,m}$ and $\mathcal{H}_{n,m}|_{\psi_\lambda}$have the specialised subsets $\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_\lambda}$, $\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_\lambda}$ which are still well defined through the specialisation and moreover they describe two bases using the remarks (1) and 2)).
From this, we get that:

$$\varphi^{\hat{W}}_{n,m}(\beta)|_{\eta_\lambda} = \underset{\mathscr{B}_{\hat{W}_{n,m}}|_{\eta_\lambda}}{\varphi^{\hat{W}^{N-1}}_{n,m}(\beta)}$$

$$l_{n,m}(\beta)|_{\psi_\lambda} = \underset{\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_\lambda}}{l_{n,m}|_{\psi_\lambda}(\beta)}$$

Combining the previous two conclusions, 5) and 6) we obtain that:

$$\underset{\mathscr{B}_{\hat{W}^{\lambda}_{n,m}}}{\varphi^{\hat{W}^{N-1}}_{n,m}(\beta)} = \underset{\mathscr{B}_{\mathcal{H}_{n,m}}|_{\psi_\lambda}}{l_{n,m}|_{\psi_\lambda}(\beta)}$$

This shows that the braid group actions $\varphi^{\hat{W}^{N-1}}_{n,m}$ and $l_{n,m}|_{\psi_\lambda}$ are isomorphic. □

Definition 1.5.3.3. *1) Let us denote B^{or}_n to be the set of oriented braids on n strands.*
2) For $\beta \in B_{2n}$, we call oriented plat closure to be a way of closing the braid to a link by cupping all the upper strands in pairs $(i, i+1)$ with caps oriented to the right and all the bottom strands in pairs $(i, i+1)$ with cups oriented to the left.

Lemma 1.5.3.4. *Let L be an oriented link. Then there exist $n \in \mathbb{N}$ and $\beta_{2n} \in B_{2n}$ such that it leads to the link by oriented plat closure:*

$$L = \hat{\hat{\beta}}^{or}_{2n}$$

Proof. There is known that there exists an oriented braid $\beta^{or}_n \in B^{or}_n$ such that $L = \hat{\beta}^{or}$. Then, we pick the first strand of L that continues as a trivial circle outside of the braid. We move the straight part such that it arrives between the first and second strand of β^{or}_n. We continue with the trivial part of the second strand, and pull it over the braid, until it arrives between the strands which were initially second and third. We continue the algorithm inductively. We obtain L as a plat closure of a braid, but the orientation of cups and caps can be in any way. If we see a cup oriented to the right, then we add a twist on top of it and transform it to be left-oriented. We do the same for caps. Finally we get the desired β_{2n}. □

1.6 Homological model for the Coloured Jones Polynomial

In this section, we present a geometric model for the Coloured Jones polynomials. We will start with a link and consider a braid that leads to the link by plat closure. Firstly we will study the Reshetikhin-Turaev functor on a link diagram that leads to the invariant, by separating it on three main levels. Secondly, we will describe step by step for each of these levels a homological counterpart using the Lawrence representation and its dual. Finally, we will show that the evaluation of the Reshetikhin-Turaev functor on the whole link corresponds to the geometric intersection pairing between the homological counterparts.

Consider the colour $N \in \mathbb{N}$. Let the parameter $\lambda = N-1$ and the specialisations as in Section 1.5:

$$\eta_{N-1} \qquad \psi_{N-1}.$$

For the simplicity of the notations, we denote:

$$\hat{W}_{n,m} := \hat{W}^{N-1}_{n,m} \qquad W_{n,m} := W^{N}_{n,m}$$

$$\hat{\varphi}^{\hat{W}}_{n,m} := \varphi^{\hat{W}^{N-1}}_{n,m} \qquad \varphi^{W}_{n,m} := \varphi^{W^{N}}_{n,m}$$

We recall the change of coefficients from 1.4.5:

$$\alpha_\lambda : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$$

defined by the formula $\alpha_\lambda(x) = q^{2\lambda}$; $\alpha_\lambda(d) = -q^{-2}$.

Theorem 1.6.0.1. *(***Homological model for coloured Jones polynomials***)* *Let $n \in \mathbb{N}$. Then, for any colour $N \in \mathbb{N}$ there exist two homology classes*

$$\mathscr{F}^N_n \in H_{2n,n(N-1)}|_{\alpha_{N-1}} \quad and \quad \mathscr{G}^N_n \in H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$$

such that for any link L for which there exists $\beta_{2n} \in B_{2n}$ with $L = \hat{\hat{\beta}}^{or}_{2n}$ (oriented plat closure 3.1.0.8), the N^{th} coloured Jones polynomial has the formula:

$$J_N(L,q) = < \beta_{2n}\mathscr{F}^N_n, \mathscr{G}^N_n > |_{\alpha_{N-1}}$$

Proof. Let L be a link and $\beta_{2n} \in B_{2n}$ such that $L = \hat{\hat{\beta}}^{or}_{2n}$ (oriented plat closure). Consider the corresponding planar diagram for the link L, using the plat closure of the braid β_{2n} which has three main levels:
1) the upper closure with n caps (oriented to the right) $\cap \cap \cap \cap$
2) the braid β_{2n}
3) the lower closure with n cups (oriented to the left) $\cup \cup \cup \cup$

Step I Following the definition of the Reshetikhin-Turaev construction, the Coloured Jones polynomial 1.2.4.1, can by obtained in the following way:

$$J_N(L,q) = \left(\overrightarrow{\mathrm{ev}}^{\otimes n}_{V_N} \circ \mathbb{F}_{V_N}(\beta_{2n}) \circ \overleftarrow{\mathrm{coev}}^{\otimes n}_{V_N} \right)(1) \in \mathbb{Z}[q^{\pm 1}]$$

As we have seen in section 1.5, quantum representations of the braid group have homological information. Therefore, for the braid part we are interested to have the action on $V_N^{\otimes 2n}$ but there is a difference with the orientation of the strands. In the previous picture, corresponding to the braid group action, we have the Reshetikhin-Turaev functor:

$$\mathbb{F}_{V_N}(\beta_{2n}) \in Aut((V_N^* \otimes V_N)^{\otimes n})$$

We will compose at the first and the third level with isomorphisms that transform $(V_N)^*$ into V_N and back, Then, at the middle level we will have the Reshetikhin-Turaev functor between V_N^{2n}, which is exactly the quantum representation $\varphi_2^{V_N} n$. Let us make this precise:

Lemma 1.6.0.2. *Let V, W finite dimensional representations of $\mathscr{U}$ and $\phi : V \to V^*$ isomorphism of representations. Then there is the following commutation relation:*

$$R_{V^* \otimes W} = (Id \otimes \phi) \circ R_{V,W} \circ (\phi^{-1} \otimes Id)$$

Proof.

$$R_{V^* \otimes W} = \mathscr{R}_{V^* \otimes W} \circ \tau = (\mathscr{R}_{V^* \otimes W} \circ (Id \otimes \phi)) \circ \big((Id \otimes \phi^{-1}) \circ \tau\big) =$$
$$= ((Id \otimes \phi) \circ \mathscr{R}_{V \otimes W}) \circ \big(\tau \circ (\phi^{-1} \otimes Id)\big) = (Id \otimes \phi) \circ R_{V,W} \circ (\phi^{-1} \otimes Id)$$

□

Denote β^{or}_{2n} the braid with the topological support β_{2n} and the orientation inherited from the oriented plat closure. Also let $\tilde{\beta}^{or}_{2n}$ be the same braid where we put all the strands to be oriented upwards. The previous lemma tell us the following:

Corollary 1.6.0.3.

$$\mathbb{F}_{V_N}(\beta_{2n}^{or}) = \left((Id_{V_N} \otimes \alpha_N)^{\otimes n}\right) \circ \mathbb{F}_{V_N}(\tilde{\beta}_{2n}^{or}) \left(Id_{V_N} \otimes \alpha_N^{-1})^{\otimes n}\right)$$

We remark that for the diagram, we need the specific orientation for the braid, but through the Reshetikhin-Turaev functor it is enough to know the source and the target and that encodes the orientation:

$$\mathbb{F}_{V_N}(\tilde{\beta}_{2n}^{or}) = \varphi_{2n}^{V_N}(\beta_{2n})$$

Using the previos two formulas, the first remark from Step I and 1.2, we conclude:

$$J_N(L,q) = \left(\overrightarrow{\mathrm{Ev}}_{V_N}^{\otimes n} \circ \varphi_{2n}^{V_N}(\beta_{2n}) \circ \overleftarrow{\mathrm{Coev}}_{V_N}^{\otimes n}\right)(1) \ \in \mathbb{Z}[q^{\pm 1}]$$

$$\begin{tikzcd}
 & & \mathbb{Z}[q^{\pm}] & & & & & \mathbb{Z}[q^{\pm}] \\
\mathbf{V}_N^{\otimes n} \arrow[urr, "1)\ \overrightarrow{\mathrm{Ev}}_{V_N}^{\otimes n}"] & \supseteq & \mathbf{W}_{2n,n(N-1)} \arrow[u, "2)\ \overrightarrow{\mathrm{Ev}}_{V_N}^{\otimes n}"'] & \subseteq & \widehat{\mathbf{W}}_{2n,n(N-1)} \arrow[ull, "3)\overrightarrow{\mathrm{Ev}}_{\hat{V}_{\eta_{N-1}}}^{\otimes n}"'] & \simeq_{\Theta_{N-1}} & \mathbf{H}_{2n,n(N-1)}|_{\psi_{N-1}} & \otimes\ \mathbf{H}^{\partial}_{2n,n(N-1)}|_{\psi_{N-1}} \arrow[u, "\langle\,,\rangle_{\psi}"'] \\
\mathbf{V}_N^{\otimes n} \arrow[u, "1)\varphi_{2n}^{V_N}(\beta)"] \arrow[ur, phantom, "\mathbf{Rk\ 1}\equiv"] & \supseteq & \mathbf{W}_{2n,n(N-1)} \arrow[u, "2)\varphi^{W}_{2n,n(N-1)}(\beta)"'] \arrow[ur, phantom, "\mathbf{Rk\ 2}\equiv"] & \subseteq_{\iota} & \widehat{\mathbf{W}}_{2n,n(N-1)} \arrow[u, "3)\hat{\varphi}^{\hat{W}}_{2n,n(N-1)}(\beta)"'] \arrow[ur, phantom, "\mathbf{Th.Kohno}\equiv"] & \simeq_{\Theta_{N-1}} & \mathbf{H}_{2n,n(N-1)}|_{\psi_{N-1}} \arrow[u, "4)l_{2n,n(N-1)}(\beta)"'] & \\
 & & \mathbb{Z}[q^{\pm}] \arrow[ull, "1)\ \overleftarrow{\mathrm{Coev}}_{V_N}^{\otimes n}"] \arrow[u, "2)\ \overleftarrow{\mathrm{Coev}}_{V_N}^{\otimes n}\ 3)"'] & & & & &
\end{tikzcd}$$

Step II The important remark is the fact that, from algebraic properties, by following the first morphism $\overleftarrow{\mathrm{Coev}}_{V_N}^{\otimes n}$, we naturally arrive a particular highest weight space.

Remark 1.6.0.4. $$Im(\overleftarrow{\mathrm{Coev}}_{V_N}^{\otimes n}) \subseteq W_{2n,n(N-1)} \qquad (\subseteq V_N^{\otimes n})$$

Proof. From the fact that $\overleftarrow{\text{Coev}}\colon \mathbb{Z}[q^{\pm}] \to V_N^{\otimes 2}$ is an isomorphism $\mathscr{U}$-modules, this will commute with the E and K-actions. Since $\mathbb{Z}[q^{\pm 1}]$ is regarded as being the trivial representation, this shows that:

$$Im\left(\overleftarrow{\text{Coev}}^{\otimes n}\right) \subseteq Ker(E \curvearrowright V_N^{\otimes 2n}) \cap Ker\left((K - Id) \curvearrowright V_N^{\otimes 2n}\right)$$

From this, one gets that for any vector $v \in Im\left(\overleftarrow{\text{Coev}}_{V_N}^{\otimes n}\right)$:

$$Kv = v = q^0 v$$

After writing $q^0 = q^{\mathbf{2n}(N-1)-2(\mathbf{N-1})\mathbf{n}}$, we conclude that:

$$Im(\overleftarrow{\text{Coev}}_{V_N}^{\otimes n}) \subseteq W_{2n,n(N-1)}$$

□

We know that the action $B_{2n} \curvearrowright V_N^{\otimes 2n}$ preserves the highest weight spaces, in particular preserves $W_{2n,n(N-1)}$. Using this invariance, we notice that actually we can obtain $J_N(L,q)$ using the highest weight spaces, by composing the morphisms from the second column (2)).

Remark 1.6.0.5.

$$J_N(L,q) = \left(\overrightarrow{\text{Ev}}_{V_N}^{\otimes n} \circ \varphi^W_{2n,n(N-1)}(\beta_{2n}) \circ \overleftarrow{\text{Coev}}_{V_N}^{\otimes n}\right) \tag{1}$$

Problem 1.6.0.6. *The highest weight spaces $W_{2n,m}$ of the finite dimensional module $V_N^{\otimes 2n}$, do not have a geometric counterpart known yet. This is one of the reasons why there are not known geometric interpretations for these invariants. On the other hand, by Kohno's Theorem, we know a geometric flavor of the bigger highest weight spaces $\hat{W}_{2n,n(N-1)}$, which live inside the power of the Verma module $\hat{V}_{N-1}$.*

Step III Having this in mind, we will look at the inclusion

$$W_{2n,n(N-1)} \hookrightarrow_{\iota} \hat{W}_{2n,n(N-1)}$$

In this part, we will study the behaviour of this inclusion with respect to the braid group action. More precisely, we will see that if we start from the small highest weight spaces, and see them as subsets of the big ones, then these are left invariant by the braid group action.

Lemma 1.6.0.7. *The B_n-action on the highest weight spaces from the Verma module, lives invariant the highest weight spaces of the finite dimensional module:*

$$\hat{\varphi}^{\hat{W}}_{n,m}\mid_{W_{n,m}}=\varphi^{W}_{n,m} \qquad \forall n,m\in\mathbb{N}$$

Proof. In 1.2.3.3, it is given the action of the R-matrix on $\hat{V}\otimes\hat{V}$. We are interested in this action, specialised with the parameter $\lambda = N-1$ and $s=q^{\lambda}$.
It follows that $R \curvearrowright \hat{V}_{N-1}\otimes\hat{V}_{N-1}$ has the form:

$$R(\hat{v}_i\otimes\hat{v}_j)=q^{-(N-1)(i+j)}\sum_{n=0}^{i}F_{i,j,n}(q)\prod_{k=0}^{n-1}(q^{(N-1)-k-j}-q^{-((N-1)-k-j)})\,\hat{v}_{j+n}\otimes\hat{v}_{i-n}$$

Firstly, we will prove that this action preserves $V_N\otimes V_N$, inside $\hat{V}_{N-1}\otimes\hat{V}_{N-1}$ (1.2.2.3). We will show this, by checking it on the basis $\{\hat{v}_i\otimes\hat{v}_j|0\leq i,j\leq N-1\}$. Let $0\leq i,j\leq N-1$. In the formula above, all the second components $\hat{v}_{i-n}$ will remain in V_N. For the first components, suppose that we pass over V_N, and $j+n\geq N$. The idea is that in this situation, the coefficient will vanish.
We notice that $q^{(N-1)-k-j}-q^{-((N-1)-k-j)}=0$ if $k=N-1-j$.
If $j+n\geq N$, it follows that $N-1-j\leq n-1$, so the term corresponding to $k=N-1-j$ will appear in the previous product, so the coefficient of $v_{j+n}\otimes v_{i-n}$ vanishes.

Secondly, we will look at the action: $B_n\curvearrowright W_{n,m}\subseteq\hat{W}_{n,m}$.
We will show that each generator σ_i preserves $W_{n,m}$.
Let $w\in W_{n,m}$. Using that $W_{n,m}\subseteq V_{n,m}$, it is possible to write

$$w=\sum_{e\in E^N_{n,m}}\alpha_e\hat{v}_{e_1}\otimes\hat{v}_{e_i}\otimes\hat{v}_{e_{i+1}}\otimes...\otimes\hat{v}_{e_n}$$

We have that $\sigma_i w=(Id^{i-1}\otimes R\otimes Id^{n-i-1})w$.
From the first part, $\sigma_i w$ will modify just the components i and $i+1$ of w, and always the indexes of those vectors will remain smaller than N. This shows that:

$$\sigma_i w\in V_N^{\otimes n}. \qquad (a)$$

Since the action of B_n is an action over $U_{\mathbf{q}}$-modules, this commutes with the action of E and K, so it preserves the weights and the kernel of E. (b)

As a conclusion, from (a) and (b), $\sigma_i w \in \hat{W}_{n,m} \cap V_N^{\otimes n}$ and so $\sigma_i w \in W_{n,m}$. $\square$

Step IV The strategy will be to use the highest weight spaces from the Verma module in the algebraic description of the coloured Jones polynomial.

We remember that the evaluation $\overrightarrow{\mathrm{Ev}}_{\hat{V}_{N-1}}$ on the Verma module $\hat{V}_{N-1}^{\otimes 2}$ restricted to $V_N^{\otimes 2}$ gives the evaluation $\overrightarrow{\mathrm{Ev}}_{V_N}(1.2)$. From this, we can obtain the invariant starting with the co-evaluation on the highest weight spaces $W_{2n,n(N-1)}$ from V_N, then following the inclusion into $\hat{W}_{2n,n(N-1)}$, the braid group action on those and finally close with the evaluation on the "big" highest weight spaces $\overrightarrow{\mathrm{Ev}}_{\hat{V}_{N-1}}$ (following the column 3)). We conclude that the coloured Jones polynomial has the form:

$$J_N(L,q) = \left(\overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{N-1}} \circ \hat{\varphi}^{\hat{W}}_{2n,n(N-1)}(\beta_{2n}) \circ \iota \circ \overleftarrow{\mathrm{Coev}}^{\otimes n}_{V_N} \right) \tag{2}$$

Step V Now, we will pas towards homological classes. The advantage of the "big" highest weight spaces $\hat{W}_{2n,n(N-1)}$ is the fact that they have an homological correspondent, given by the Lawrence representation $\mathcal{H}_{2n,n(N-1)}$, due to Kohno's relation. We will consider the element corresponding to the image of 1 through the co-evaluation, followed by inclusion, which lives in $\hat{W}_{2n,n(N-1)}$. After that, we will reverse it to the geometrical part given by the Lawrence construction, using Khono's function.

Definition 1.6.0.8. *Let $v \in \hat{W}_{2n,n(N-1)}$ described by:*

$$v = \iota \circ \overleftarrow{\mathrm{Coev}}^{\otimes n}_{V_N}(1) \in \hat{W}_{2n,n(N-1)}$$

$$\mathscr{F}_0 := \Theta^{-1}_{N-1}(v) \in \mathcal{H}_{2n,n(N-1)}|\psi_{N-1}$$

Remark 1.6.0.9. *From Kohno's Theorem, we obtain the following identification:*

$$\hat{\varphi}^{\hat{W}}_{2n,n(N-1)}(\beta_{2n})(v) = \Theta_{N-1}\left(l_{2n,n(N-1)}|_{\psi_{N-1}}(\beta_{2n})(\mathscr{F}_0)\right)$$

In other words, we would like to stress the fact that the correspondence between v and $\mathscr{F}_0$ one to the other through the fuction Θ_{N-1}, is preserved by the action of B_{2n}.

Remark 1.6.0.10.

$$J_N(L,q) = \left(\overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{\eta_{N-1}}} \circ \hat{\varphi}^{\hat{W}}_{2n,n(N-1)}(\beta_{2n})\ (v) \right) =$$

$$1.6.0.9 = \left(\overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{\eta_{N-1}}} \circ \Theta_{N-1} \circ l_{2n,n(N-1)}|_{\psi_{N-1}}(\beta_{2n})\ (\mathscr{F}_0) \right)$$

Up to this point, we found the first homology class $\mathscr{F}_0$, such that encodes homologically the algebraic coevaluation $\overleftarrow{\mathrm{Coev}}^{\otimes n}_{V_N}$. Using this, the braid group action on the quantum side and on this class will correspond.

Step VI In the sequel, we are interested to find the second homology class $\mathscr{G}$, which will be a geometric counterpart for the evaluation $\overrightarrow{\mathrm{Ev}}^{\otimes n}_{V_N}$ defined on $W_{n,m}$. Even if we are interested to do this, in practice we will find a description for the evaluation $\overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{N-1}}$ on $\hat{W}_{2n,n(N-1)}$, but which encodes basically the evaluation on the highest spaces of the finite dimensional module. We will study $\overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{N-1}}$ as an element of the dual space $(\hat{W}_{2n,n(N-1)})^*$ and make the correspondence with a geometric element in element in $H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$. We will use the discussion from Section1.4.5 about different flovours of specialisations of the Blanchfield pairing.

We remind that we have the non-degenerate, sesquilinear pairing:

$$<,> |\alpha_{N-1} : \mathcal{H}_{2n,n(N-1)}|_{\alpha_{N-1}} \otimes \mathcal{H}^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}} \to \mathbb{Q}(q)$$

Definition 1.6.0.11. *Let* $G_0 := \overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{N-1}} \circ \Theta_{N-1} \in Hom(\mathcal{H}_{2n,n(N-1)}|_{\psi_{N-1}}, \mathbb{Z}[q^{\pm}])$.
Consider the dual element $\mathscr{G}^N_n \in \mathcal{H}^{\partial}_{n,m}|_{\alpha_{N-1}}$ *given by 1.4.5.7.*
Let $\mathscr{F}^N_n := \mathscr{F}_0 \otimes_i 1 \in \mathcal{H}^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$

Remark 1.6.0.12. *This means that* $\forall \mathscr{E} \in \mathcal{H}_{2n,n(N-1)}|_{\alpha_{N-1}}$ *we have:*

$$G_0(\mathscr{E}) =< \mathscr{E}, \mathscr{G}^N_n > |_{\alpha_{N-1}}$$

Step VII Now we will prove that we have the following model for the coloured Jones polynomial :

$$J_N(L,q) =< \beta_{2n}\mathscr{F}^N_n, \mathscr{G}^N_n >_{\alpha_{N-1}}$$

Remark 1.6.0.13. *Let the morphism of changing the coefficients p_{N-1}, as in 1.4.5. Then p_{N-1} commutes with the braid groups actions on the two specialisations of the Lawrence representations as below:*

$$\begin{tikzcd}
\underset{\beta_{2n}\mathscr{F}_0}{\mathcal{H}_{2n,n(N-1)}|_{\psi_{N-1}}} \arrow[r, "p_{N-1}"] & \underset{\beta_{2n}\mathscr{F}^N_n}{\mathcal{H}_{2n,n(N-1)}|_{\alpha_{N-1}}} \\
\underset{\mathscr{F}_0}{\mathcal{H}_{2n,n(N-1)}|_{\psi_{N-1}}} \arrow[u, "l_{2n,n(N-1)}|_{\psi_{N-1}}"] \arrow[r, "p_{N-1}"] \arrow[ur, phantom, "\equiv"] & \underset{\mathscr{F}^N_n}{\mathcal{H}_{2n,n(N-1)}|_{\alpha_{N-1}}} \arrow[u, "l_{2n,n(N-1)}|_{\alpha_{N-1}}"']
\end{tikzcd}$$

Putting all the previous steps together, we obtain the following:

$$J_N(L,q) =^{1.6.0.10} \overrightarrow{\mathrm{Ev}}^{\otimes n}_{\hat{V}_{N-1}} \circ \Theta_{N-1} \circ l_{2n,n(N-1)}|_{\psi_{N-1}}(\beta_{2n})\ (\mathscr{F}_0) =$$

$$=^{1.6.0.11} G_0 \circ l_{2n,n(N-1)}|_{\psi_{N-1}}(\beta_{2n})\ (\mathscr{F}_0) =$$

$$=^{1.6.0.131.4.5.7} G \circ l_{2n,n(N-1)}|_{\alpha_{N-1}}(\beta_{2n})\ (\mathscr{F}^N_n) =$$

$$=^{1.6.0.12} < l_{2n,n(N-1)}|_{\alpha_{N-1}}(\beta_{2n})\ (\mathscr{F}^N_n), \mathscr{G}^N_n > |_{\alpha_{N-1}}$$

Simplifying the notations, we obtain the desired interpretation:

$$J_N(L,q) = < \beta_{2n}\ \mathscr{F}^N_n, \mathscr{G}^N_n > |_{\alpha_{N-1}}$$

which concludes the proof.

□

1.7 Topological model with non-specialised Homology classes

For the homological model for $J_N(L,q)$, we have constructed homology classes

$$\mathscr{F}^N_n \in H_{2n,n(N-1)}|_{\alpha_{N-1}} \text{and } \mathscr{G}^N_n \in H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}$$

that give the invariant by geometric pairing. We notice that the colour N appears in two places. Firstly it showes up in the number of points from the configuration space, because we are taking the configuration spaces in a fixed punctured disk (with $2n$ points removed), but with $n(N-1)$ points. Secondly, we use the spacialisation α_{N-1}, which depends on the color N.

In this section, we will show that actually $\mathscr{F}_n^N$ and $\mathscr{G}_n^N$ come from two homology classes that live in the unspecialised Lawrence representation. More precisely, they live in the homology of $\tilde{C}_{2n.n(N-1)}$, where we basically just increase the ring of coefficients. The feature of this model is that now the color shows up just in the number of points from the configuration space, but not anymore in the specialisation. In the sequel, we will prove the following statement:

Theorem 1.7.0.1. *(***Homological model for coloured Jones polynomials***)*

Let $n \in \mathbb{N}$. Then, for any colour $N \in \mathbb{N}$ there exist two homology classes

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_{\gamma} \quad \textit{and} \quad \tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\gamma}$$

such that for any link L for which there exists $\beta_{2n} \in B_{2n}$ with $L = \hat{\hat{\beta}}{}^{or}_{2n}$ (oriented plat closure 3.1.0.8), the N^{th}coloured Jones polynomial has the formula:

$$J_N(L,q) = < \beta_{2n}\tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

1.7.1 Identifications with q, s indeterminates

In the previous section 1.5, we have studied identifications between quantum representations and homological representations specialised with two complex generic parameters or to a natural number and an indeterminate. In this section, we will show that, if we increase a bit the ring of coordinates, the identification holds also over a ring with two indeterminates.

We recall that the quantum representation $\hat{W}$ is defined over $\mathbb{Z}[q^{\pm}, s^{\pm}]$. On the other hand, the Lawrence representation $\mathcal{H}_{n,m}$ is defined over $\mathbb{Z}[x^{\pm}, d^{\pm}]$. We have the following spacialisations:

$$\eta_\lambda : \mathbb{Z}[q^{\pm 1}, s^{\pm 1}] \to \mathbb{Z}[q^{\pm}]$$

$$\eta_\lambda(q) = q; \quad \eta_\lambda(s) = q^{\lambda}$$

$$\psi_\lambda : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}]$$

$$\psi_\lambda(x) = q^{2\lambda}; \quad \psi_\lambda(d) = -q^{-2}.$$

Definition 1.7.1.1. *Consider the specialisation which increase the ring of coefficients in the following manner:*

$$\xi : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[s^{\pm}, q^{\pm}]$$

$$\xi(x) = s^2; \quad \xi(d) = -q^{-2}.$$

For an argument which will appear, we will need to work over a field. Consider the inclusion:

$$j : \mathbb{Z}[s^{\pm}, q^{\pm}] \to \mathbb{Q}(s, q).$$

Then, let us define the extension of the initial ring by:

$$\gamma : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(s, q)$$

$$\gamma = j \circ \xi$$

Concerning the field that we used for the model from 1.6.0.1, let us consider the specialisation:

$$\delta_\lambda : \mathbb{Q}(s, q) \to \mathbb{Q}(q)$$

$$\delta_\lambda(s) = q^\lambda$$

Remark 1.7.1.2. *This shows that we have the following commutative diagrams between the previous specialisations of the coefficients:*

$$\begin{tikzcd}
\mathbb{Z}[x^{\pm1}, d^{\pm1}] \arrow[rr, "\gamma"] \arrow[dr, "\xi"] \arrow[ddr, "\psi_\lambda"] \arrow[dddr, "\alpha_\lambda"'] & & \mathbb{Q}(s, q) \arrow[dddl, "\delta_\lambda"] \\
& \mathbb{Z}[s^{\pm1}, q^{\pm1}] \arrow[ur, "j"] \arrow[d, "\eta_\lambda"] & \\
& \mathbb{Z}[q^{\pm1}] \arrow[d, "i"] & \\
& \mathbb{Q}(q) &
\end{tikzcd}$$

Using a similar argument as the one that we discussed in 1.5.3, one can concludes that the identification between quantum and homological representations works over a ring in two inteterminates. This was also briefly discussed in [41].

Theorem 1.7.1.3. *The braid group representations over* $\mathbb{Z}[s^{\pm}, q^{\pm}]$ *are isomorphic:*

$$\hat{W}_{n,m} \simeq_{\Theta} \mathcal{H}_{n,m}|_{\xi}$$

$$\mathscr{B}_{\hat{W}_{n,m}} \qquad \mathscr{B}_{\mathcal{H}_{n,m}}|_{\xi}$$

1.7.2 Lift of the homology classes $\mathscr{F}_n^N$ and $\mathscr{G}_n^N$

Our aim is to lift the homology classes, which leave a priori in the specialisation α_{N-1} towards two elements constructed from the Lawrence representation spacialised using the specialisation ξ. However, since in our arguments we need to work over a field in order to be able to interpret the non-degeneracy of the Blanchfield pairing by dual elements, we will use the specialisation γ.

Lemma 1.7.2.1. *There exist two homology classes*

$$\tilde{\mathscr{F}}_n^N \in H_{2n,n(N-1)}|_{\gamma} \quad \textit{and} \quad \tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\gamma}$$

such that under the specialisation δ_{N-1} *one has*

$$\tilde{\mathscr{F}}_n^N|_{\delta_{N-1}} = \mathscr{F}_n^N$$

$$\tilde{\mathscr{G}}_n^N|_{\delta_{N-1}} = \mathscr{G}_n^N.$$

Proof. 1) We remark that in the formulas from 1.2.3.4, we can use the same functions over the ring $\mathbb{Z}[s^{\pm}, q^{\pm}]$ in the definition of evaluation and coevaluation. We do not need to have the property that α_N is an isomorphism over the quantum group over this new ring, all what we need are the formulas for the two functions. Consider the vector

$$v^s = \iota^s \circ \overleftarrow{\mathrm{Coev}}^{s\otimes n}_{V_N}(1) \in \hat{W}_{2n,n(N-1)}(\text{ over}\mathbb{Z}[q^{\pm}, s^{\pm}])$$

We remark that also in 1.2.6.2, the function Θ is defined over the ring with two parameters. Using 1.7.1.3, we can consider:

$$\tilde{\mathscr{F}_0}_n^N := \Theta^{-1}(v^s) \in \mathcal{H}_{2n,n(N-1)}|\xi$$

2) On the other hand, the evaluation from 1.2.3.4 that corresponds to the caps from the diagram of the link, can be defined over the ring $\mathbb{Z}[s^{\pm}, q^{\pm}]$ in

two parameters as well. Secondly, the Blanchfield pairing will remain non-degenerate when we specialise the coefficients using the function γ. Then, dualising this evaluation and extending the coefficients to the field $\mathbb{Q}(s^{\pm}, q^{\pm})$ (as in the main proof from the previous section before), we get an homology class:

$$\tilde{\mathscr{G}}_n^N \in H^{\partial}_{2n,n(N-1)}|_{\gamma}$$

3) Let us define: $\tilde{\mathscr{F}}_n^N := \tilde{\mathscr{F}}_{0n}^{\ N}|_j$. Then, from the previous remarks we get that:

$$\mathscr{F}_n^N := \tilde{\mathscr{F}}_n^N|_{\delta_{N-1}}$$

$$\mathscr{G}_n^N := \tilde{\mathscr{G}}_n^N|_{\delta_{N-1}}$$

which concludes the proof. Moreover, the pairings are related one with the other

$$< \beta \mathscr{F}_n^N, \mathscr{G}_n^N > = < \beta \tilde{\mathscr{F}}_n^N, \tilde{\mathscr{G}}_n^N > |_{\delta_{N-1}}$$

$\square$

Corollary 1.7.2.2. *Following the homological model from 1.6.0.1 and the result concerning the lift of the homology classes 1.7.2.1, we conclude the topological model for $J_N(L,q)$, as it is presented in Theorem 1.7.0.1.*

Chapter 2

Modified Turaev-Viro Invariants from quantum $sl(2|1)$

This chapter is a joint project with Nathan Geer where we constructed quantum invariants for 3-manifolds from super quantum groups at roots of unity. This appeared as a paper on arxiv [8].

The category of finite dimensional module over the quantum superalgebra $U_q(\mathfrak{sl}(2|1))$ is not semi-simple and the quantum dimension of a generic $U_q(\mathfrak{sl}(2|1))$-module vanishes. This vanishing happens for any value of q (even when q is not a root of unity). These properties make it difficult to create a fusion or modular category. Loosely speaking, the standard way to obtain such a category from a quantum group is: 1) specialize q to a root of unity; this forces some modules to have zero quantum dimension, 2) quotient by morphisms of modules with zero quantum dimension, 3) show the resulting category is finite and semi-simple. In this chapter we show an analogous construction works in the context of $U_q(\mathfrak{sl}(2|1))$ by replacing the vanishing quantum dimension with a modified quantum dimension. In particular, we specialize q to a root of unity, quotient by morphisms of modules with zero modified quantum dimension and show the resulting category is generically finite semi-simple. Moreover, we show the categories from this chapter are relative G-spherical categories. As a consequence we obtain invariants of 3-manifold with additional structures.

2.1 Context

The numerical $6j$-symbols associated with the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ were first introduced in theoretical physics by Eugene Wigner in 1940 and Giulio (Yoel) Racah in 1942. These symbols have been generalized to quantum $6j$-symbols coming from tensor categories. If the category is fusion and spherical then the quantum $6j$-symbols lead to Turaev-Viro invariants of 3-manifold (see [20, 10, 53, 82, 81]). The prototype of such a topological invariant arises from a particular category of modules over the quantum algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$.

Let us describe the T-V construction for this example. Without modification the category of finite dimensional modules over $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is not fusion. If q is generic then there are an infinite number of non-isomorphic simple modules. When q is a root of unity then the quantum dimension of some of these modules becomes zero. Loosely speaking, by taking the quotient of such modules one obtains a category with a finite number of simple modules. More precisely, taking the quotient of the category of $U_q(\mathfrak{sl}_2(\mathbb{C}))$-modules by negligible morphisms one obtains the desired spherical and fusion category $\mathscr{S}$. Here a morphism $f : V \to W$ is negligible if for all morphisms $g : W \to V$ we have

$$Tr_q(f \circ g) = 0$$

where Tr_q is the quantum trace. If V is a simple module whose quantum dimension $\mathrm{qdim}(V) = Tr_q(\mathrm{Id}_V)$ is zero then any morphism to or from V is negligible; such a module is called negligible. The simple modules which are not negligible are said to be in the alcove.

The Turaev-Viro invariant is defined as a certain state sum computed on an arbitrary triangulation of a 3-manifold. The state sum on a triangulation $\mathcal{T}$ of a closed 3-manifold M is defined, roughly speaking, as follows: Consider states of $\mathcal{T}$ which are maps from the edges of $\mathcal{T}$ to a finite index set I corresponding to isomorphism class of simple objects in the category $\mathscr{S}$. Given a state $\varphi : \{\text{edges of } \mathcal{T}\} \to I$ one associates with each tetrahedron T of $\mathcal{T}$ a particular quantum $6j$-symbol denoted by $|T|_\varphi$. The state sum is defined by taking the product of these symbols over all tetrahedra of $\mathcal{T}$ and summing up the resulting products (with certain weights) over all I colorings of $\mathcal{T}$:

$$TV(\mathcal{T}) = \mathcal{D}^{-2v} \sum_{\varphi \text{ state}} \left(\prod_{e\in\{\text{edges of } \mathcal{T}\}} \mathrm{qdim}(\varphi(e)) \right) \left(\prod_{T\in\{\text{tetrahedron of } \mathcal{T}\}} |T|_\varphi \right) \tag{2.1}$$

where $\mathrm{qdim}(\varphi(e))$ is the quantum dimension of the simple module associated to $\varphi(e)$ and $\mathcal{D}^2 = \sum_{i\in I} \mathrm{qdim}(i)^2$.

The main point of this construction is that the state sum $TV(\mathcal{T})$ is independent of the choice of triangulation. This can be verified in two steps. First, the quantum $6j$-symbols satisfy the symmetries of the tetrahedron. Second, any two triangulations of a closed 3-manifold can be transformed into one another by a finite sequence of the so called Pachner moves and an ambient isotopy (see [73]). Thus, it is enough to check that the state sum is invariant under the Pachner moves. For the category $\mathscr{S}$, these moves correspond to well-known algebraic identities which the quantum $6j$-symbols satisfy.

Obstructions to applying this construction to a general pivotal tensor category $\mathscr{C}$ include:

1. zero quantum dimensions,
2. non-semi-simplicity of $\mathscr{C}$,
3. infinitely many isomorphism classes of simple objects of $\mathscr{C}$.

Kashaev [47] and later Baseilhac and Benedetti [12] considered 3-manifold invariants arising from a category with such obstructions, namely the category of modules over the Borel subalgebra of quantum $\mathfrak{sl}(2)$ at a root of unity. Geer, Patureau-Mirand, and Turaev [34] gave an alternate general approach to dealing with these obstructions and defined a secondary Turaev-Viro invariant of oriented 3-manifolds M. This is accomplished by three main modifications of the T-V invariant.

First, to address obstruction (1) they replace the vanishing qdim and $|T|_\varphi$ in Equation (2.1) with corresponding non-zero modified quantum dimension and modified $6j$-symbol, see [33, 34].

The second modification is dealing with the last two obstructions. If the usual state sum described above is applied to a category with infinitely many isomorphism classes of simple objects, this sum is of course infinite. With this in mind, the authors of [34] required that the pivotal tensor category $\mathscr{C}$

had additional structure, including a G-grading on $\mathscr{C}$ where G is an abelian group. To overcome the infinite sum problem, a finite number of modules are selected using a cohomology class in $H^1(M, G)$. This step also addresses obstruction (2) by requiring that generically graded pieces of the category are semi-simple.

The final modification is to introduce a link L in the manifold M. If one applies the changes described above, in the first two steps the invariant can still be zero or not well defined (see [34]). In particular, the sum of the modified dimensions over a graded piece is zero, i.e. the analogous quantity associated to $\mathcal{D}^2$ in Equation (2.1) is zero. To construct a non-zero invariant one can consider triangulations $\mathcal{T}$ of M that realized the isotopy class of L as a so called Hamiltonian link in $\mathcal{T}$ (see [12]). The Hamiltonian link is used to modify the weights in the terms of the state sum.

Let us be more precise. The construction of the modified TV-invariant works in the context of a relative G-spherical category (see Section 2.2 for details): Let G be an abelian group with a small symmetric subset $\mathcal{X} \subset G$. Let $\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$ be a G-graded pivotal $\Bbbk$-category such that

1. $\mathscr{C}_g$ is finitely semi-simple for each $g \in G \setminus \mathcal{X}$,
2. there exists a t-ambi pair $(\mathsf{A}, \mathsf{d} : \mathsf{A} \to \Bbbk^\times)$ where each object of A is isomorphic to a unique element of $\cup_{g \in G \setminus \mathcal{X}} \mathcal{I}_g$ and $\mathcal{I}_g$ represents the isomorphic classes of simple objects of $\mathscr{C}_g$,
3. there exists a map $\mathsf{b} : \mathsf{A} \to \Bbbk$ satisfying the condition in Definition 2.2.3.1.

In [34] it is shown that a relative G-spherical category (with basic data) gives rise to modified quantum $6j$-symbols. In this context, these $6j$-symbols are not numbers but rather tensors on certain multiplicity spaces.

Let M be a closed orientable 3-manifold and L a link in M. Following [12], we consider H-triangulation $(\mathcal{T}, \mathcal{L})$ of (M, L): $\mathcal{T}$ is a quasi-regular triangulation of M, $\mathcal{L}$ is a set of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$ and the union of the edges in $\mathcal{L}$ is the link L. Let $\Phi : \{\text{edges}\} \to G$ be a G-valued 1-cocycle on $\mathcal{T}$ which takes values in $G \setminus \mathcal{X}$. A *state* of the G-coloring Φ is a map φ assigning to every oriented edge e of $\mathcal{T}$ an element $\varphi(e)$ of $\mathcal{I}_{\Phi(e)}$ such that $\varphi(-e) = \varphi(e)^*$ for all e. The set of all states of Φ is denoted $\mathrm{St}(\Phi)$.

Give a state φ and a tetrahedron T of $\mathcal{T}$ we can associate a modified $6j$-symbols $|T|_\varphi$, for details see [34]. Any face of $\mathcal{T}$ belongs to exactly two tetrahedra of $\mathcal{T}$, and the associated multiplicity modules are dual to each other. The tensor product of the $6j$-symbols $|T|_\varphi$ associated to all tetrahedra T of $\mathcal{T}$ can be contracted using this duality. We denote by cntr the tensor product of all these contractions. Let $\mathcal{T}_1$ be the set of unoriented edges $\mathcal{T}$ and let $\mathcal{T}_3$ the set of tetrahedra of $\mathcal{T}$. Set

$$TV(\mathcal{T}, \mathcal{L}, \Phi) = \sum_{\varphi \in \mathrm{St}(\Phi)} \prod_{e \in \mathcal{T}_1 \setminus \mathcal{L}} \mathsf{d}(\varphi(e)) \prod_{e \in \mathcal{L}} \mathsf{b}(\varphi(e)) \ \ \mathrm{cntr}\left(\bigotimes_{T \in \mathcal{T}_3} |T|_\varphi \right) \in \Bbbk.$$

Theorem 2.1.0.1. *$TV(\mathcal{T}, \mathcal{L}, \Phi)$ depends only on the isotopy class of L in M and the cohomology class $[\Phi] \in H^1(M, G)$. It does not depend on the choice of the H-triangulation of (M, L) and on the choice of Φ in its cohomology class.*

In this paper we show that the category $\mathscr{D}$ of finite dimensional modules over the quantum linear Lie superalgebra $U_q(\mathfrak{sl}(2|1))$ leads to a relative G-spherical category and so gives rise to a modified TV-invariant. Our approach is based on a generalization of the usual technique: we take a quotient by negligible morphisms corresponding to the modified trace. As we will discuss, the obtained invariants should have different properties than the usual TV-invariants.

The category $\mathscr{D}$ has the obstructions (1)–(3) listed above. When q is generic, the simple $U_q(\mathfrak{sl}(2|1))$-modules are indexed by pairs (n, α) where $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Even when q is generic most of these simple modules have vanishing quantum dimensions. Therefore, taking the quotient of $\mathscr{D}$ by the negligible morphisms corresponding to the quantum trace would trivialize most of the category. Alternatively, the ideal $\mathcal{I}$ generated by the four dimensional module $V(0, \alpha)$ has a non-zero modified trace and corresponding non-zero modified quantum dimensions. When $q = e^{2\pi i/l}$ is a root of unity some of these modified quantum dimensions become zero. In particular, the modified quantum dimensions of modules on the boundary of the "alcove" vanish. The idea behind this paper is to take the quotient of $\mathscr{D}$ by these modules to obtain a relative G-spherical category.

This construction has a novel feature: The alcove has an infinite number of non-isomorphic simple modules $V(n, \tilde{\alpha})$ where $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$, $0 \leq n \leq l' - 2$ and $l' = \frac{2l}{3+(-1)^l}$. Let $V(n, \alpha)$ be the simple module over $U_q(\mathfrak{sl}(2|1))$ when q is generic. By setting $q = e^{2\pi i/l}$ the module $V(n, \alpha)$ specializes to $V(n, \alpha + l\mathbb{Z})$ for $\alpha \in \mathbb{C}$. Modified $6j$-symbols corresponding to these modules exist for both q generic and $q = e^{2\pi i/l}$. An interesting problem is to see if the $6j$-symbols coming from generic q will specialize to the $6j$-symbols constructed in this paper for roots of unity of any order. If this is true one could see if there exist an invariant defined for generic q which when specialized to $q = e^{2\pi i/l}$, for any l recovers the modified TV-invariants corresponding to the relative G-spherical categories defined in this paper. Results in this direction have been obtain for the usual Reshetikhin-Turaev quantum invariants, see [37, 38, 61, 70] and references within.

What is different in our context is that the modified invariants arising from the work of this paper require manifolds with additional structures, including the cohomology class discussed above. This makes the definition of the invariants more technical but can also add new information. For example, the modified RT-type invariants of [23, 21] recover the multivariable Alexander polynomial and Reidemeister torsion, which allows the reproduction of the classification of lens spaces. Also, the invariants of [23, 21] give rise to TQFTs and mapping class group representations with the notable property that the action of a Dehn twist has infinite order. This is in strong contrast with the usual quantum mapping class group representations where all Dehn twists have finite order. We expect similar properties for the invariants coming from this paper. Combining such properties with the ideas discussed in the previous paragraph could lead to appealing applications.

We should also mention that 3-manifold invariants arising from a quantization of $\mathfrak{sl}(2|1)$ have already been constructed by Ha in [36]. Ha's construction uses a "unrolled" version of quantum $\mathfrak{sl}(2|1)$. He also uses a modified trace on a different ideal which consists of projective modules, most of which only exist when q is a root of unity. In comparison, in this paper the quotient of $\mathscr{D}$ we take contains all the projective modules of $\mathscr{D}$. Also, for $0 \leq n \leq l'-2$ and $\alpha \in \mathbb{C}$ the ideal Ha works with does not contain a module which is the specialization of the simple $U_q(\mathfrak{sl}(2|1))$-module $V(n, \alpha)$ discussed above.

2.2 Categorical Preliminaries

As mentioned above our main theorem will be that $U_q(\mathfrak{sl}(2|1))$ gives rises to a relative G-spherical category. With this in mind, in this section we will recall the general definition of such a category, for more details see [31, 34].

2.2.1 $\Bbbk$-categories

Let $\Bbbk$ be a field. A tensor $\Bbbk$-category is a tensor category $\mathscr{C}$ such that its hom-sets are left $\Bbbk$-modules, the composition and tensor product of morphisms is $\Bbbk$-bilinear, and the canonical $\Bbbk$-algebra map $\Bbbk \to \mathrm{End}_{\mathscr{C}}(\mathbb{I}), k \mapsto k\ \mathrm{Id}_{\mathbb{I}}$ is an isomorphism (where $\mathbb{I}$ is the unit object). A tensor category is pivotal if it has dual objects and duality morphisms

$$\overrightarrow{\mathrm{coev}}_V \colon \mathbb{I} \to V \otimes V^*, \quad \overrightarrow{\mathrm{ev}}_V \colon V^* \otimes V \to \mathbb{I}, \quad \overleftarrow{\mathrm{coev}}_V \colon \mathbb{I} \to V^* \otimes V \quad \text{and} \quad \overleftarrow{\mathrm{ev}}_V \colon V \otimes V^* \to \mathbb{I}$$

which satisfy compatibility conditions (see for example [20, 27]).

An object V of $\mathscr{C}$ is simple if $\mathrm{End}_{\mathscr{C}}(V) = \Bbbk\,\mathrm{Id}_V$. Let V be an object in $\mathscr{C}$ and let $\alpha : V \to W$ and $\beta : W \to V$ be morphisms. The triple (V, α, β) (or just the object V) is a retract of W if $\beta\alpha = \mathrm{Id}_V$. An object W is a direct sum of the finite family $\{V_i\}_i$ of objects of $\mathscr{C}$ if there exist retracts (V_i, α_i, β_i) of W with $\beta_i\alpha_j = 0$ for $i \neq j$ and $\mathrm{Id}_W = \sum_i \alpha_i\beta_i$. An object which is a direct sum of simple objects is called semi-simple.

2.2.2 Colored ribbon graph invariants

Let $\mathscr{C}$ be a pivotal $\Bbbk$-category. A ribbon graph is formed from several oriented framed edges colored by objects of $\mathscr{C}$ and several coupons colored with morphisms of $\mathscr{C}$. We say a $\mathscr{C}$-colored ribbon graph in $\mathbb{R}^2$ (resp. $S^2 = \mathbb{R}^2 \cup \{\infty\}$) is called planar (resp. spherical). Let F be the usual Reshetikhin-Turaev functor from the category of $\mathscr{C}$-colored planar ribbon graphs to $\mathscr{C}$ (see [81]).

Let $T \subset S^2$ be a closed $\mathscr{C}$-colored ribbon graph. Let e be an edge of T colored with a simple object V of $\mathscr{C}$. Cutting T at a point of e, we obtain a $\mathscr{C}$-colored ribbon graph T_V in $\mathbb{R} \times [0,1]$ where $\mathsf{F}(T_V) \in \mathrm{End}(V) = \Bbbk\,\mathrm{Id}_V$. We call T_V a cutting presentation of T and let $\langle T_V \rangle \in \Bbbk$ denote the isotopy invariant of T_V defined from the equality $\mathsf{F}(T_V) = \langle T_V \rangle\ \mathrm{Id}_V$.

Let A be a class of simple objects of $\mathscr{C}$ and $\mathsf{d} : \mathsf{A} \to \Bbbk^\times$ be a mapping such that $\mathsf{d}(V) = \mathsf{d}(V^*)$ and $\mathsf{d}(V) = \mathsf{d}(V')$ if V is isomorphic to V'. We say (A, d) is t-ambi pair if for any closed $\mathscr{C}$-colored trivalent ribbon graph T with any two cutting presentations T_V and $T_{V'}$, $V, V' \in \mathsf{A}$ the following equation holds:

$$\mathsf{d}(V)\langle T_V \rangle = \mathsf{d}(V')\langle T'_V \rangle.$$

2.2.3 G-graded and generically G-semi-simple categories

Let G be a group. A pivotal $\Bbbk$-category is G-graded if for each $g \in G$ we have a non-empty full subcategory $\mathscr{C}_g$ of $\mathscr{C}$ such that

1. $\mathbb{I} \in \mathscr{C}_e$, (where e is the identity element of G)
2. $\mathscr{C} = \bigoplus_{g\in G} \mathscr{C}_g$,
3. if $V \in \mathscr{C}_g$, then $V^* \in \mathscr{C}_{g^{-1}}$,
4. if $V \in \mathscr{C}_g$, $V' \in \mathscr{C}_{g'}$ then $V \otimes V' \in \mathscr{C}_{gg'}$,
5. if $V \in \mathscr{C}_g$, $V' \in \mathscr{C}_{g'}$ and $\mathrm{Hom}_{\mathscr{C}}(V, V') \neq 0$, then $g = g'$.

For a subset $\mathcal{X} \subset G$ we say:

1. $\mathcal{X}$ is symmetric if $\mathcal{X}^{-1} = \mathcal{X}$,
2. $\mathcal{X}$ is small in G if the group G can not be covered by a finite number of translated copies of $\mathcal{X}$, in other words, for any $g_1, \dots, g_n \in G$, we have $\bigcup_{i=1}^{n}(g_i\mathcal{X}) \neq G$.

1. A $\Bbbk$-category $\mathscr{C}$ is semi-simple if all its objects are semi-simple.
2. A $\Bbbk$-category $\mathscr{C}$ is finitely semi-simple if it is semi-simple and has finitely many isomorphism classes of simple objects.
3. A G-graded category $\mathscr{C}$ is a generically finitely G-semi-simple category if there exists a small symmetric subset $\mathcal{X} \subset G$ such that for each $g \in G \setminus \mathcal{X}$, $\mathscr{C}_g$ is finitely semi-simple. By a generic simple object we mean a simple object of $\mathscr{C}_g$ for some $g \in G \setminus \mathcal{X}$.

The notion of generically G-semi-simple categories appears in [34, 32] through the following generalization of fusion categories (in particular, fusion categories satisfy the following definition when G is the trivial group, $\mathcal{X} = \emptyset$ and $\mathsf{d} = \mathsf{b} = \mathrm{qdim}_{\mathscr{C}}$ is the quantum dimension):

Definition 2.2.3.1 (Relative G-spherical category)**.** *Let $\mathscr{C}$ be a generically finitely G-semi-simple pivotal $\Bbbk$-category with small symmetric subset $\mathcal{X} \subset G$ and let A be the class of generic simple objects of $\mathscr{C}$. We say that $\mathscr{C}$ is* $(\mathcal{X}, \mathsf{d})$-relative G-spherical *if*

1. *there exists a map* $\mathsf{d} : \mathsf{A} \to \Bbbk^\times$ *such that* (A, d) *is a t-ambi pair,*

2. *there exists a map* $\mathsf{b} : \mathsf{A} \to \Bbbk$ *such that* $\mathsf{b}(V) = \mathsf{b}(V^*)$*,* $\mathsf{b}(V) = \mathsf{b}(V')$ *for any isomorphic objects* $V, V' \in \mathsf{A}$ *and for any* $g_1, g_2, g_1g_2 \in G \setminus \mathcal{X}$ *and* $V \in G_{g_1g_2}$ *we have*

$$\mathsf{b}(V) = \sum_{V_1 \in irr(\mathscr{C}_{g_1}),\, V_2 \in irr(\mathscr{C}_{g_2})} \mathsf{b}(V_1)\, \mathsf{b}(V_2) \dim_{\Bbbk}(\mathrm{Hom}_{\mathscr{C}}(V, V_1 \otimes V_2))$$

where $irr(\mathscr{C}_{g_i})$ *denotes a representing set of isomorphism classes of simple objects of* $\mathscr{C}_{g_i}$*.*

If $\mathscr{C}$ is a category with the above data, for brevity we say $\mathscr{C}$ is a relative G-spherical category. In [34], to construct a 3-manifold invariant from a relative G-spherical category $\mathscr{C}$ the authors assume that $\mathscr{C}$ has a technical requirement called basic data. The following lemma (proved in [31]) says that in most situations $\mathcal{X}$ can be enlarged so that $\mathscr{C}$ has basic data. This lemma implies that we can assume the categories we considered in this paper have basic data.

Lemma 2.2.3.2. *If no object of* A *is isomorphic to its dual, then* $\mathscr{C}$ *contains a basic data. In particular, basic data exists if* $\mathcal{X}$ *contains the set* $\{g \in G : g = g^{-1}\}$*.*

2.2.4 Traces on ideals in pivotal categories

In this subsection we recall some facts about right traces in a pivotal $\Bbbk$-category $\mathscr{C}$, for more details see [35, 27]. In this paper we will use right traces to show that a t-ambi pair exists.

By a right ideal of $\mathscr{C}$ we mean a full subcategory $\mathcal{I}$ of $\mathscr{C}$ such that:

1. If $V \in \mathcal{I}$ and $W \in \mathscr{C}$, then $V \otimes W \in \mathcal{I}$.

2. If $V \in \mathcal{I}$ and if $W \in \mathscr{C}$ is a retract of V, then $W \in \mathcal{I}$.

A right trace on a right ideal $\mathcal{I}$ is a family of linear functions

$$\{\mathsf{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to \Bbbk\}_{V \in \mathcal{I}}$$

such that:

1. If $U, V \in \mathcal{I}$ then for any morphisms $f : V \to U$ and $g : U \to V$ in $\mathscr{C}$ we have
$$\mathsf{t}_V(gf) = \mathsf{t}_U(fg).$$
2. If $U \in \mathcal{I}$ and $W \in \mathscr{C}$ then for any $f \in \mathrm{End}_{\mathscr{C}}(U \otimes W)$ we have
$$\mathsf{t}_{U\otimes W}(f) = \mathsf{t}_U\left((\mathrm{Id}_U \otimes \overleftarrow{\mathrm{ev}}_W)(f \otimes \mathrm{Id}_{W^*})(\mathrm{Id}_U \otimes \overrightarrow{\mathrm{coev}}_W)\right)$$

Next we recall how to construct a right trace. Given an object V of $\mathscr{C}$ we define the ideal generated by V as
$$\mathcal{I}_V = \{W \in \mathscr{C} \mid W \text{ is a retract of } V \otimes X \text{ for some object } X\}.$$
In [35] the notion of a right ambidextrous simple object is developed (see Sections 4.2 and 4.4 of [35]). Theorem 10 of [35] implies:

Theorem 2.2.4.1 ([35])**.** *If V is a right ambidextrous simple object then there exists a non-zero right trace $\{\mathsf{t}_V\}$ on the ideal $\mathcal{I}_V$; this trace is unique up to multiplication by a non-zero scalar.*

Now we will recall a way to show a simple object is right ambidextrous. Let V be a simple object in $\mathscr{C}$. We fix a direct sum decomposition of $V \otimes V^*$ into indecomposable objects W_i indexed by a set I:
$$V \otimes V^* = \bigoplus_{k\in I} W_k. \tag{2.2}$$
Let $i_k : W_k \to V \otimes V^*$ and $p_k : V \otimes V^* \to W_k$ be the morphisms corresponding to this decomposition. In particular, $\sum_{k\in I} i_k p_k = \mathrm{Id}_{V\otimes V^*}$ and $p_k i_k = \mathrm{Id}_{W_k}$, for all $k \in I$. From Lemma 3.1.1 of [27] we have the following lemma:

Lemma 2.2.4.2 ([27])**.** *There exists unique $j, j' \in I$ so that*

1. *$\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, W_j)$ is non-zero and is spanned by $p_j \overrightarrow{\mathrm{coev}}_V$ and*
2. *$\mathrm{Hom}_{\mathscr{C}}(W_{j'}, \mathbb{I})$ is non-zero and is spanned by $\overleftarrow{\mathrm{ev}}_V\, i_{j'}$.*

Theorem 3.1.3. of [27] gives the following theorem.

Theorem 2.2.4.3 ([27])**.** *The simple object V is right ambidextrous if and only if $j = j'$.*

Finally, let us explain how to produce a t-ambi pair from a right trace. Let t be a right trace on a right ideal $\mathcal{I}$ of $\mathscr{C}$. Let d be the modified dimension associated with t defined by $\mathsf{d}(V) = \mathsf{t}_V(\mathrm{Id}_V)$ for $V \in \mathcal{I}$. Set

$$\mathsf{B} = \{V \in \mathcal{I} \cap \mathcal{I}^* \,|\, V \text{ is simple and } \mathsf{d}(V) = \mathsf{d}(V^*)\}$$

where $\mathcal{I}^* = \{V \in \mathscr{C} \mid V^* \in \mathcal{I}\}$. The following theorem is Corollary 7 in [35].

Theorem 2.2.4.4 ([35])**.** *The pair* (B, d) *is a t-ambi pair.*

2.3 Quantum $\mathfrak{sl}(2|1)$ at roots of unity

2.3.1 Notation

Fix a positive integer $l \geq 3$ and let $q = e^{\frac{2\pi\sqrt{-1}}{l}}$ be a l^{th}-root of unity. Set

$$l' = \begin{cases} l & \text{if } l \text{ is odd} \\ l/2 & \text{if } l \text{ is even} \end{cases}.$$

We use two quotients of the complex numbers: $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}/l\mathbb{Z}$. We will use greek letters to denote elements of $\mathbb{C}$. We will denote elements in $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}/l\mathbb{Z}$ with bars and tildes, respectively. In this paper, both $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}/l\mathbb{Z}$ are abelian groups induced from the addition in $\mathbb{C}$.

For $\alpha \in \mathbb{C}$, let $\tilde{\alpha}$ be the element of $\mathbb{C}/l\mathbb{Z}$ such that α is in the equivalence class of $\tilde{\alpha}$. In other words, α maps to $\tilde{\alpha}$ under the obvious map $\mathbb{C} \to \mathbb{C}/l\mathbb{Z}$. Similarly, for $\alpha \in \mathbb{C}$ or $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ let $\bar{\alpha} \in \mathbb{C}/\mathbb{Z}$ such that α or $\tilde{\alpha}$ is mapped to $\bar{\alpha}$ under the map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ or $\mathbb{C}/l\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$, respectively. For x in $\mathbb{C}$ or $\mathbb{C}/l\mathbb{Z}$ set $\{x\} = q^x - q^{-x}$ and $[x] = \frac{\{x\}}{\{1\}}$.

2.3.2 Superspaces

A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_{|0|} \oplus V_{|1|}$ over $\mathbb{C}$. We denote the parity of a homogeneous element $x \in V$ by $|x| \in \mathbb{Z}_2$. We say x is even (odd) if $x \in V_{|0|}$ (resp. $x \in V_{|1|}$). If V and W are $\mathbb{Z}_2$-graded vector spaces then the space of linear maps $\mathrm{Hom}_{\mathbb{C}}(V, W)$ has a natural $\mathbb{Z}_2$-grading given by $f \in \mathrm{Hom}_C(V, W)_{|j|}$ if $f(V_{|i|}) \subset W_{|i|+|j|}$ for $|i|, |j| \in \mathbb{Z}_2$. Throughout, all modules of over a $\mathbb{Z}_2$-graded ring will be $\mathbb{Z}_2$-graded modules.

2.3.3 The superalgebra $U_q(\mathfrak{sl}(2|1))$

Let $A = (a_{ij})$ be the 2×2 matrix given by $a_{11} = 2$, $a_{12} = a_{21} = -1$ and $a_{22} = 0$. Let $U_q(\mathfrak{sl}(2|1))$ be the $\mathbb{C}$-superalgebra generated by the elements K_i, K_i^{-1}, E_i and F_i, $i = 1, 2$, subject to the relations:

$$K_i^{\pm 1} K_j = K_j K_i^{\pm 1}, \qquad K_i^{-1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{-1}, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1^2 = 0, \quad F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1^2 = 0 \tag{2.3}$$

where $[,]$ is the super-commutator given by $[x, y] = xy - (-1)^{|x||y|} yx$. All generators are even except for E_2 and F_2 which are odd. The algebra $U_q(\mathfrak{sl}(2|1))$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + K_i^{-1} \otimes E_i, & \epsilon(E_i) &= 0 & S(E_i) &= -K_i E_i \\
\Delta(F_i) &= F_i \otimes K_i + 1 \otimes F_i, & \epsilon(F_i) &= 0 & S(F_i) &= -F_i K_i^{-1} \\
\Delta(K_i) &= K_i \otimes K_i & \varepsilon(K_i) &= 1, & S(K_i) &= K_i^{-1}, \\
\Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1} & \varepsilon(K_i^{-1}) &= 1, & S(K_i^{-1}) &= K_i.
\end{aligned}$$

2.3.4 Representations of $U_q(\mathfrak{sl}(2|1))$

A $U_q(\mathfrak{sl}(2|1))$-module V is a weight module if V is finite dimensional and both K_1 and K_2 act diagonally on V. Let $\mathscr{D}$ be the tensor category of finite dimensional $\mathbb{Z}_2$-graded weight $U_q(\mathfrak{sl}(2|1))$-modules, whose unit object is the trivial module $\mathbb{I} = \mathbb{C}$. It is easy to see that this category is a $\mathbb{C}$-category.

A direct calculation shows that $S^2(x) = K_2^{-2} x K_2^2$ for all $x \in U_q(\mathfrak{sl}(2|1))$. Thus, the square of the antipode is equal the conjugation of a group-like element and so $\mathscr{D}$ is a pivotal category (see [14, Proposition 2.9]). In particular, for any object V in $\mathscr{D}$, the dual object and the duality morphisms are defined as follows: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$\begin{aligned}
\overrightarrow{\mathrm{coev}}_V &: \mathbb{C} \to V \otimes V^* \text{ is given by } 1 \mapsto \sum v_j \otimes v_j^*, \\
\overrightarrow{\mathrm{ev}}_V &: V^* \otimes V \to \mathbb{C} \text{ is given by } f \otimes w \mapsto f(w), \\
\overleftarrow{\mathrm{coev}}_V &: \mathbb{C} \to V^* \otimes V \text{ is given by } 1 \mapsto \sum (-1)^{|v_j|} v_j^* \otimes K_2^2 v_j, \\
\overleftarrow{\mathrm{ev}}_V &: V \otimes V^* \to \mathbb{C} \text{ is given by } v \otimes f \mapsto (-1)^{|f||v|} f(K_2^{-2} v),
\end{aligned} \tag{2.4}$$

where $\{v_j\}$ is a basis of V and $\{v_j^*\}$ is the dual basis of V^*. These morphisms satisfy the compatibility conditions of a pivotal category.

The simple $U_q(\mathfrak{sl}(2|1))$-modules have been studied in [1]. Here we will consider what they call the typical type A representations: let $\omega \in \{\pm 1\}$, $0 \leq n \leq l' - 1$ and $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ then there exists a highest weight module $V(\omega, n, \tilde{\alpha})$ with highest weight vector v such that

$$E_i v = 0, \quad K_1 v = \omega q^n v \quad \text{and} \quad K_2 v = q^{\tilde{\alpha}} v.$$

In [1] it is shown that under certain conditions $V(\omega, n, \tilde{\alpha})$ is a simple module of dimension $4(n+1)$. Let us now give these conditions. To simplify notation if $\omega = 1$ we set $V(n, \tilde{\alpha}) = V(1, n, \tilde{\alpha})$.

Proposition 2.3.4.1 ([1], page 873)**.** *If $[\tilde{\alpha}] \cdot [\tilde{\alpha} + n + 1] \neq 0$ then $V(n, \tilde{\alpha})$ is simple.*

Remark 2.3.4.2. *We use slightly different notation than [1]. Our module $V(n, \tilde{\alpha})$ corresponds to the module from [1, page 873] with the following parameters:*

$$\omega = 1, \ N = n + 1, \ \lambda_1 = q^n, \ \mu_1 = n = N - 1, \ \lambda_2 = q^{\alpha}, \ \mu_2 = \alpha.$$

Since

$$[x] = 0 \Leftrightarrow \frac{q^x - q^{-x}}{q - q^{-1}} = 0 \Leftrightarrow q^x - q^{-x} = 0 \Leftrightarrow q^{2x} = 1 \Leftrightarrow x \in \frac{l}{2}\mathbb{Z}$$

then the above proposition implies that $V(n, \tilde{\alpha})$ is simple if $\tilde{\alpha} \notin (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z}) - (n + 1))/l\mathbb{Z}$. In particular, if $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\} \subseteq \mathbb{C}/\mathbb{Z}$ then $V(n, \tilde{\alpha})$ is simple.

Theorem 2.3.4.3 ([1])**.** *Let $n \in \{0, ..., l' - 1\}$ and $\tilde{\alpha} \notin (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z}) - (n + 1))/l\mathbb{Z}$. Then $V(n, \tilde{\alpha})$ has a basis $\{w_{\rho,\sigma,p} | p \in \{0, ..., n\}; \rho, \sigma \in \{0, 1\}\}$ whose action of $U_q(\mathfrak{sl}(2|1))$ is given by:*

$$K_1 \cdot w_{\rho,\sigma,p} = q^{\rho-\sigma+n-2p} w_{\rho,\sigma,p}, \qquad K_2 \cdot w_{\rho,\sigma,p} = q^{\tilde{\alpha}+\sigma+p} w_{\rho,\sigma,p}, \tag{2.5}$$

$$F_1 \cdot w_{\rho,\sigma,p} = q^{\sigma-\rho} w_{\rho,\sigma,p+1} - \rho(1 - \sigma) q^{-\rho} w_{\rho-1,\sigma+1,p}, \quad F_2 \cdot w_{\rho,\sigma,p} = (1 - \rho) w_{\rho+1,\sigma,p}, \tag{2.6}$$

$$E_1 \cdot w_{\rho,\sigma,p} = -\sigma(1 - \rho) q^{n-2p+1} w_{\rho+1,\sigma-1,p} + [p][n - p + 1] w_{\rho,\sigma,p-1}, \tag{2.7}$$

$$E_2 \cdot w_{\rho,\sigma,p} = \rho[\tilde{\alpha} + p + \sigma] w_{\rho-1,\sigma,p} + \sigma(-1)^{\rho} q^{-\tilde{\alpha}-p} w_{\rho,\sigma-1,p+1}. \tag{2.8}$$

Here the super grading of this basis is given by $|w_{\rho,\sigma,p}| = \rho + \sigma \in \mathbb{Z}/2\mathbb{Z}$.

Since $w_{1,1,n}$ is a lowest weight vector of $V(n,\tilde{\alpha})$ with weight $(-n,\tilde{\alpha}+\tilde{n}+\tilde{1})$ then we have

$$V(n,\tilde{\alpha})^* = V(n,-\tilde{\alpha}-\tilde{n}-\tilde{1}).$$

We will use the modules of the form $V(0,\tilde{\alpha})$ extensively. With this in mind we highlight the structure of such modules. If $\tilde{\alpha} \notin (\frac{l}{2}\mathbb{Z})/l\mathbb{Z} \cup ((\frac{l}{2}\mathbb{Z})-(n+1))/l\mathbb{Z}$ then $V(0,\tilde{\alpha})$ is a 4-dimensional module with the following action:

	$w_{0,0}$ $\rho=0,\sigma=0$	$w_{1,0}$ $\rho=0,\sigma=0$	$w_{0,1}$ $\rho=0,\sigma=0$	$w_{1,1}$ $\rho=0,\sigma=0$
K_1	$w_{0,0}$	$qw_{1,0}$	$q^{-1}w_{0,1}$	$w_{1,1}$
K_2	$q^{\tilde{\alpha}}w_{0,0}$	$q^{\tilde{\alpha}}w_{1,0}$	$q^{\tilde{\alpha}+1}w_{0,1}$	$q^{\tilde{\alpha}+1}w_{1,1}$
E_1	0	0	$w_{1,0}$	0
E_2	0	$[\tilde{\alpha}]w_{0,0}$	0	$[\tilde{\alpha}+1]w_{0,1}$
F_1	0	$w_{0,1}$	0	0
F_2	$w_{1,0}$	0	$w_{1,1}$	0

We will use the following lemma in the proof of the Decomposition Lemma below.

Lemma 2.3.4.4. *Let V be an object in $\mathscr{C}$. Suppose $V_1,...,V_n$ are simple submodules of V such that V_i is not isomorphic to V_j for all $i \neq j$ and*

$$\dim(V_1)+...+\dim(V_n)=\dim(V).$$

Then $V = V_1 \oplus ... \oplus V_n$*.*

Proof. Consider the statement $P(k)$: if $i_1, ..., i_k, j \in \{1, ..., n\}$ are all different then

$$V_j \cap (V_{i_1} + ... + V_{i_k}) = \{0\}.$$

If $P(k)$ were true for $k \in \{1, ..., n-1\}$ then $V_1 \oplus ... \oplus V_n$ would be a submodule of V. The hypothesis on the dimensions would then imply

$$V = V_1 \oplus ... \oplus V_n.$$

Thus, it suffices to prove $P(k)$ holds for $k \in \{1, ..., n-1\}$. We will do this by induction.

First, we will show $P(1)$ holds. Let $i, j \in \{1, ..., n\}$ such that $V_i \cap V_j \neq \{0\}$ and $i \neq j$. Therefore, there is a non-zero vector in $V_i \cap V_j$ which generates a submodule W of $V_i \cap V_j$. In particular, W is a submodule of both V_i and V_j. Since these modules are simple we have W is equal to both V_i and V_j. Thus, $V_i = V_j$ which is a contradiction.

Next assuming $P(k)$ is true we will show $P(k+1)$ holds. Let $i_1, ..., i_{k+1}$ and j be unique elements of $\{1, ..., n\}$. Suppose by contradiction that $V_j \cap (V_{i_1} + ... + V_{i_{k+1}}) \neq \{0\}$ and let v be a non-zero vector in this intersection. Let $< v >$ be the non-zero module generated by v. Since $v \in V_j$ then $< v >$ is a submodule of V_j. But V_j is simple so $< v >= V_j$. Similarly, $v \in V_{i_1} + ... + V_{i_{k+1}}$ implying $< v >$ is in this sum and we conclude

$$V_j \subseteq V_{i_1} + ... + V_{i_{k+1}}.$$

From the induction step for $P(k)$, we deduce that $V_{i_1} + ... + V_{i_{k+1}} = V_{i_1} \oplus ... \oplus V_{i_{k+1}}$. Combining the last two observations, we have V_j is a submodule of $V_{i_1} \oplus ... \oplus V_{i_{k+1}}$. This implies $\mathrm{Hom}_{\mathscr{C}}(V_j, V_{i_1} \oplus ... \oplus V_{i_{k+1}}) \neq \{0\}$ since the inclusion morphism is in this space.

On the other hand, since the simple modules V_j and V_{i_s} are non-isomorphic for $s \in \{1, .., k+1\}$ then $\mathrm{Hom}_{\mathscr{C}}(V_j, V_{i_s}) = \{0\}$. This implies

$$\mathrm{Hom}_{\mathscr{C}}(V_j, V_{i_1} \oplus ... \oplus V_{i_{k+1}}) = \mathrm{Hom}_{\mathscr{C}}(V_j, V_{i_1}) \oplus ... \oplus \mathrm{Hom}_{\mathscr{C}}(V_j, V_{i_{k+1}}) = \{0\}.$$

But above we showed this homomorphism space was non-zero so we have a contradiction. Thus, the induction step is complete. □

Lemma 2.3.4.5. *(Decomposition Lemma) Let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\alpha}, \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$ and $\bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Then for any $n \in \{0, ..., l'-2\}$ we have*

$$V(0,\tilde{\alpha}) \otimes V(n,\tilde{\beta}) = V(n,\tilde{\alpha}+\tilde{\beta}) \oplus V(n+1,\tilde{\alpha}+\tilde{\beta}) \oplus V(n-1,\tilde{\alpha}+\tilde{\beta}+1) \oplus V(n,\tilde{\alpha}+\tilde{\beta}+1)$$

where we use the convention that $V(-1, \tilde{\alpha}+\tilde{\beta}+1) = 0$ in the case when $n = 0$.

Proof. We will prove the case when $n \neq 0$ (the case n=0 will be analogous). Since $\bar{\alpha}, \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, it means that $V(0,\tilde{\alpha})$ and $V(n,\tilde{\beta})$ have the structure described in Theorem 2.3.4.3: let $\{w^{(0,\tilde{\alpha})}_{\rho,\sigma}\}$ and $\{w^{(n,\tilde{\beta})}_{\rho,\sigma,p}\}$ be the bases of $V(0,\tilde{\alpha})$ and $V(n,\tilde{\beta})$, respectively.

We will prove that there are four highest weight vectors:

$$v_{(n,\tilde{\alpha}+\tilde{\beta})}, v_{(n+1,\tilde{\alpha}+\tilde{\beta})}, v_{(n-1,\tilde{\alpha}+\tilde{\beta}+1)}, v_{(n,\tilde{\alpha}+\tilde{\beta}+1)} \in V(0,\tilde{\alpha}) \otimes V(n,\tilde{\beta})$$

where the weight of $v_{(i,\tilde{\gamma})}$ is $(q^i, q^{\tilde{\gamma}})$. First, clearly $v_{(n,\tilde{\alpha}+\tilde{\beta})} := w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{0,0,0}$ is a highest weight vector of weight $(n, \tilde{\alpha}+\tilde{\beta})$. Second, we want to find a highest weight vector $v_2 = v_{(n+1,\tilde{\alpha}+\tilde{\beta})}$ with weight $(q^{n+1}, q^{\tilde{\alpha}+\tilde{\beta}})$. We'll search for v_2 as a combination of the form:

$$v_2 = w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{1,0,0} + c \cdot w^{(0,\tilde{\alpha})}_{1,0} \otimes w^{(n,\tilde{\beta})}_{0,0,0}.$$

To find c we check that E_1 and E_2 act by zero. For any c we have $E_1(v_2) = 0$. On the other hand, $E_2(v_2) = 0$ implies $c = -q^{-\tilde{\alpha}} \cdot \frac{[\tilde{\beta}]}{[\tilde{\alpha}]}$. So,

$$v_2 = w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{1,0,0} - q^{-\tilde{\alpha}} \cdot \frac{[\tilde{\beta}]}{[\tilde{\alpha}]} \cdot w^{(0,\tilde{\alpha})}_{1,0} \otimes w^{(n,\tilde{\beta})}_{0,0,0}$$

is a highest weight vector.

Third, we want a highest weight vector $v_3 = v_{(n-1,\tilde{\alpha}+\tilde{\beta}+1)}$ of the form

$$c_1 w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{0,1,0} + c_2 w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{1,0,1} + c_3 w^{(0,\tilde{\alpha})}_{1,0} \otimes w^{(n,\tilde{\beta})}_{0,0,1} + c_4 w^{(0,\tilde{\alpha})}_{0,1} \otimes w^{(n,\tilde{\beta})}_{0,0,0}.$$

After checking the conditions which come from the action of E_1 and E_2, and setting $c_2 = 1$ we obtain:

$$\begin{aligned} v_3 = q^{-(n+1)}[1][n] \cdot w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{0,1,0} + w^{(0,\tilde{\alpha})}_{0,0} \otimes w^{(n,\tilde{\beta})}_{1,0,1} \\ - \frac{1}{[\tilde{\alpha}]}(q^{-(\tilde{\alpha}+\tilde{\beta}+n+1)}[1][n] + q^{-\tilde{\alpha}}[\tilde{\beta}+1]) \cdot w^{(0,\tilde{\alpha})}_{1,0} \otimes w^{(n,\tilde{\beta})}_{0,0,1} \\ - \frac{q^{-1}[1][n]}{[\tilde{\alpha}]}(q^{-(\tilde{\alpha}+\tilde{\beta}+n+1)})[1][n] + q^{-\tilde{\alpha}}[\tilde{\beta}+1]) \cdot w^{(0,\tilde{\alpha})}_{0,1} \otimes w^{(n,\tilde{\beta})}_{0,0,0}. \end{aligned}$$

Similarly we obtain:

$$v_4 = v_{(n,\tilde{\alpha}+\tilde{\beta}+1)} = -q^{-\tilde{\alpha}}\frac{[\tilde{\alpha}]}{[\tilde{\beta}+1]}w_{0,0}^{(0,\tilde{\alpha})}\otimes w_{1,1,0}^{(n,\tilde{\beta})}$$
$$+ q^{-\tilde{\alpha}-1}\frac{[\tilde{\beta}]}{[\tilde{\alpha}+1]}(q^{-n}+q^{-1-\tilde{\beta}}\frac{[1][n]}{[\tilde{\beta}+1]})w_{1,1}^{(0,\tilde{\alpha})}\otimes w_{0,0,0}^{(n,\tilde{\beta})}$$
$$+ (q^{-n}+q^{-1-\tilde{\beta}}\frac{[1][n]}{[\tilde{\beta}+1]})w_{0,1}^{(0,\tilde{\alpha})}\otimes w_{1,0,0}^{(n,\tilde{\beta})}$$
$$+ w_{1,0}^{(0,\tilde{\alpha})}\otimes w_{0,1,0}^{(n,\tilde{\beta})} - \frac{q^{-\tilde{\beta}}}{[\tilde{\beta}+1]}w_{1,0}^{(0,\tilde{\alpha})}\otimes w_{1,0,1}^{(n,\tilde{\beta})}.$$

Consider the submodule $W_{(i,\tilde{\gamma})}$ of $V(0,\tilde{\alpha})\otimes V(n,\tilde{\beta})$ generated by one of the highest weight vectors $v_{(i,\tilde{\gamma})}$ constructed above. As mentioned above the classification of $U_q(sl(2|1))$-highest weight modules is given in [1]. From this classification, since $W_{(i,\tilde{\gamma})}$ is a highest weight module of weight $(q^i, q^{\tilde{\gamma}})$, with $\bar{\gamma} = \bar{\alpha}+\bar{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ it follows that $W_{(i,\tilde{\gamma})}$ is isomorphic to $V(i,\tilde{\gamma})$ and is a simple of dimension $4(i+1)$.

Thus, we have

$$\dim(W_{(n,\tilde{\alpha}+\tilde{\beta})}) + \dim(W_{(n+1,\tilde{\alpha}+\tilde{\beta})}) + \dim(W_{(n-1,\tilde{\alpha}+\tilde{\beta}+1)}) + \dim(W_{(n,\tilde{\alpha}+\tilde{\beta}+1)})$$
$$= 4((n+1)+(n+2)+n+(n+1)) = 4(4n+4) = 16(n+1).$$

But $\dim(V(0,\tilde{\alpha})\otimes V(n,\tilde{\beta})) = 4\cdot 4(n+1) = 16(n+1)$. So, the four submodules satisfy the conditions of Lemma 2.3.4.4, which means that their direct sum is isomorphic to $V(0,\tilde{\alpha})\otimes V(n,\tilde{\beta})$. □

From the previous result, we obtain that, with some weight restrictions, the decomposition of the tensor product of two typical modules depends just on the total weight-sum, and it is independent on the two separate components. More precisely:

Corollary 2.3.4.6. *Consider $\tilde{\alpha},\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}, n \in \{0,...,l'-1\}$ such that $\bar{\alpha},\bar{\beta},\bar{\alpha}+\bar{\beta} \notin \{\bar{0},\overline{\frac{l}{2}}\}$. Then*

$$V(0,\tilde{\alpha})\otimes V(n,\tilde{\beta}) \simeq V(0,\widetilde{\alpha+\epsilon})\otimes V(n,\widetilde{\beta-\epsilon})$$

for any $\epsilon \in \mathbb{C}$ such that $\overline{\alpha+\epsilon},\overline{\beta-\epsilon} \notin \{\bar{0},\overline{\frac{l}{2}}\}$.

For $g \in \mathbb{C}/\mathbb{Z}$, let $\mathscr{D}_g$ be the full subcategory of $\mathscr{D}$ whose objects are all $U_q(\mathfrak{sl}(2|1))$-module V such that the central element K_2^l acts as multiplication by q^{lg}. In particular, for $0 \leq n \leq l'-1$ and $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ we have $V(n, \tilde{\alpha}) \in \mathscr{D}_{\bar{\alpha}}$. This gives a $\mathbb{C}/\mathbb{Z}$-grading on the category $\mathscr{D}$ and we write $\mathscr{D} = \bigoplus_{g \in \mathbb{C}/\mathbb{Z}} \mathscr{D}_g$.

2.3.5 The subcategory $\mathscr{C}$ of $\mathscr{D}$

Now we want to construct a subcategory $\mathscr{C}$ of $\mathscr{D}$ that will eventually (after taking a quotient) lead to our invariants for 3-manifolds. From Corollary 2.3.4.6, we see that the tensor product of two representations , doesn't change if we modify their second components in a balanced way such that they still satisfy the technical conditions of avoiding $\frac{1}{2}$. One of our main aims, is to arrive at a category which is generically semi-simple, which means semi-simple in any grading excepting few special gradings.

The idea that we have in mind is that we want to do this by a certain induction on the number of components of a tensor product of $V(0, \alpha)$. However, we need to make sure that when we take a product of $n+1$ representations, we can find two of them which satisfy the requirement of the previous Corollary. Then the idea would be to choose a good ϵ, and ballance them by it as in the formula 2.3.4.6 , such that one of them together with all the other $n-1$ representations have a good weight so that we can apply the induction hypothesis for them.

In the sequel, we will define a way of choosing this subcategory in order to make sure that for any tensor product of $V(0, \alpha)$'s from this set, we can find always two of them which satisfy the requirements 2.3.4.6.

Definition 2.3.5.1. *Set $\mathcal{Y} = (\frac{1}{4}\mathbb{Z})/\mathbb{Z}$. Let $\mathscr{C}$ the full sub-category of $\mathscr{D}$ containing the trivial module and all retracts of a module of the form*

$$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes ... \otimes V(0, \tilde{\alpha}_n) \tag{2.9}$$

where $\tilde{\alpha}_1, ..., \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha}_1, ..., \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$.

Lemma 2.3.5.2. *The category $\mathscr{C}$ is a $\mathbb{C}/\mathbb{Z}$-graded pivotal $\mathbb{C}$-category, where the grading and pivotal structure are induced from $\mathscr{D}$.*

Proof. Let W_1 and W_2 in $\mathscr{C}$. From the definition, for $j = 1, 2$, there exists $\tilde{\alpha}_{j,1}, ..., \tilde{\alpha}_{j,n_j} \in \mathbb{C}/l\mathbb{Z}$, with $\bar{\alpha}_{j,1}, ..., \bar{\alpha}_{j,n_j} \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ such that W_j be is retract of

$V_j := V(0, \tilde{\alpha}_{j,1}) \otimes ... \otimes V(0, \tilde{\alpha}_{j,n_j})$. Let $p_j : V_j \to W_j$ and $q_j : W_j \to V_j$ be the morphisms of this retract. Then $V_1 \otimes V_2$ is of the form of the module in Equation (2.9) with all $\bar{\alpha}_{j,n} \notin \mathcal{Y}$. It follows that $W_1 \otimes W_2$ is an object of $\mathscr{C}$ since it is a retract of $V_1 \otimes V_2$ with maps $p_1 \otimes p_2$ and $q_1 \otimes q_2$. Therefore, $\mathscr{C}$ is a tensor category. Moreover, $\mathscr{C}$ is $\mathbb{C}$-category since it is a full sub-category of the $\mathbb{C}$-category $\mathscr{D}$.

Finally, we will check that $\mathscr{C}$ is closed under duality. Let $W \in \mathscr{C}$. Then W is a retract of some $V := V(0, \tilde{\alpha}_1) \otimes ... \otimes V(0, \tilde{\alpha}_n)$ such that $\bar{\alpha}_1, ..., \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$. Then W^* is a retract of V^* and we have that:

$$V^* \cong (V(0, \tilde{\alpha}_1) \otimes ... \otimes V(0, \tilde{\alpha}_n))^* \cong V(0, \tilde{\alpha}_n)^* \otimes ... \otimes V(0, \tilde{\alpha}_1)^*$$

$$\cong V(0, -\tilde{\alpha}_n - \tilde{1}) \otimes ... \otimes V(0, -\tilde{\alpha}_1 - \tilde{1}).$$

But $-\bar{\alpha}_n - \bar{1}, ..., -\bar{\alpha}_1 - \bar{1} \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ so we have $W^* \in \mathscr{C}$. Thus, since $\mathscr{C}$ is a full subcategory of $\mathscr{D}$ then the duality morphisms of $\mathscr{D}$ give a pivotal structure in $\mathscr{C}$. Finally, the $\mathbb{C}/\mathbb{Z}$-grading on $\mathscr{D}$ induces a $\mathbb{C}/\mathbb{Z}$-grading on $\mathscr{C}$. □

The Decomposition Lemma 2.3.4.5 says we can decompose the tensor product $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\beta})$ into simple modules if $\bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Given a module as in Equation (2.9), the following lemma says we can always find a pair $\tilde{\alpha}_i, \tilde{\alpha}_j$ with this property. This fact is one of the motivations for the choice of the set $\mathcal{Y}$.

Lemma 2.3.5.3. *For any $\tilde{\alpha}_1, ..., \tilde{\alpha}_n \in \mathbb{C}/\mathbb{Z}$ such that $\bar{\alpha}_1, ..., \bar{\alpha}_n \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{Y}$ and*

$$\bar{\alpha}_1 + ... + \bar{\alpha}_n \notin \{\bar{0}, \frac{\bar{l}}{2}\}$$

then there exist $i, j \in \{1, ..., n\}$ such that $i \neq j$ and $\bar{\alpha}_i + \bar{\alpha}_j \notin \{\bar{0}, \frac{\bar{l}}{2}\}$.

Proof. If $n = 2$, we have just two numbers and from the hypothesis they have the desired sum.

Let us consider the case $n \geq 3$ and let suppose by contradiction that

$$\overline{\alpha_i + \alpha}_j \in \{\bar{0}, \frac{\bar{l}}{2}\}, \tag{2.10}$$

for all $i, j \in \{1, ..., n\}$ with $i \neq j$. Up to a reordering, we can suppose that there exists $m \in \{2, ..., n\}$ such that:

- $\overline{\alpha_1 + \alpha_i} = \bar{0}, \forall i \in \{2, ..., m\}$
- $\overline{\alpha_1 + \alpha_j} = \frac{\bar{l}}{2}, \forall j \in \{m+1, ..., n\}$.

This implies the following:

- $\bar{\alpha}_i = -\bar{\alpha}_1, \forall i \in \{2, ..., m\}$
- $\bar{\alpha}_i = \frac{\bar{l}}{2} - \bar{\alpha}_1, \forall j \in \{m+1, ..., n\}$.

Now we have three cases.

Case 1. If $m \geq 3$, then $\bar{\alpha}_2 = \bar{\alpha}_3 = -\bar{\alpha}_1$ which implies

$$\bar{\alpha}_2 + \bar{\alpha}_3 = -2\bar{\alpha}_1 \notin \{\bar{0}, \frac{\bar{l}}{2}\}, \quad \text{since } \bar{\alpha}_1 \notin \{\frac{\bar{1}}{2}, \frac{\bar{l}}{4}\}$$

which is a contradiction with our supposition.

Case 2. If $n - m \geq 2$, then $\bar{\alpha}_{m+1} = \bar{\alpha}_{m+2} = \frac{\bar{l}}{2} - \bar{\alpha}_1$ which implies

$$\bar{\alpha}_{m+1} + \bar{\alpha}_{m+2} = -2\bar{\alpha}_1.$$

Here as above this leads to a contradiction.

Case 3. If we are not in the first two cases and $n \neq 2$ then it means $n = 3$ and $m = 2$. In this case we have

- $\bar{\alpha}_2 = -\bar{\alpha}_1$
- $\bar{\alpha}_3 = \frac{\bar{l}}{2} - \bar{\alpha}_1$.

The relations above lead to:

$$\bar{\alpha}_2 + \bar{\alpha}_3 = \frac{\bar{l}}{2} - 2\bar{\alpha}_1.$$

But from the initial supposition, we have that $\bar{\alpha}_2 + \bar{\alpha}_3 \in \{\bar{0}, \frac{\bar{l}}{2}\}$.

If $\bar{\alpha}_2 + \bar{\alpha}_3 = \bar{0}$, it implies that $\frac{\bar{l}}{2} - 2\bar{\alpha}_1 = \bar{0}$, so $\bar{\alpha}_1 = \frac{\bar{l}}{4}$ which contradicts that $\bar{\alpha}_1 \notin \mathcal{Y} = \frac{1}{4}\mathbb{Z}/\mathbb{Z}$.

If $\bar{\alpha}_2 + \bar{\alpha}_3 = \frac{\bar{l}}{2}$, then $\frac{\bar{l}}{2} - 2\bar{\alpha}_1 = \frac{\bar{l}}{2}$, and it means $\bar{\alpha}_1 \in \{\bar{0}, \frac{\bar{1}}{2}\}$ which is impossible since $\bar{\alpha}_1 \notin \mathcal{Y}$.

Thus all cases lead to contradictions and so the lemma follows. □

The next part is devoted to an argument that will lead to the fact that the tensor product of simple modules in the alcove is commutative. The proof uses the braiding of the "un-rolled" quantum $U_q^H(\mathfrak{sl}(2|1))$, studied by Ha in [36]. In his paper he works with odd ordered roots of unity but as we observe his proof also works for even roots of unity (at least for the existence of a braiding, it may not extend to the twist).

Let $U_q^H = U_q^H(\mathfrak{sl}(2|1))$ be the superalgebra generated by the elements K_i, K_i^{-1}, h_i, E_i and F_i, $i = 1, 2$, subject to the relations in (3.1) and

$$[h_i, E_j] = a_{ij}E_j, \quad [h_i, F_j] = -a_{ij}F_j, \quad [h_i, h_j] = 0, \quad [h_i, K_j] = 0$$

for $i, j = 1, 2$. All generators are even except E_2 and F_2 which are odd. This superalgebra is a Hopf algebra where the coproduct, counit and anitpode of K_i, K_i^{-1}, E_i and F_i are given in Subsection 2.3.3 and

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \epsilon(h_i) = 0, \quad S(h_i) = -h_i$$

for $i, j = 1, 2$.

For a U_q^H-module V let $q^{h_i} : V \to V$ be the operator defined by $q^{h_i}(v) = q^{\lambda_i}v$ where v is a weight vector with respect to h_i of weight λ_i. The superalgebra ideal I generated by $E_1^{l'}$ and $F_1^{l'}$ is a Hopf algebra ideal (i.e. an ideal in the kernel of the counit, a coalgebra coideal and stable under the antipode). Let $\mathscr{D}^H$ be the category of finite dimensional U_q^H/I-modules with even morphisms such that $q^{h_i} = K_i$ as operators for $i = 1, 2$. Since U_q^H/I is a Hopf superalgebra then $\mathscr{D}^H$ is a tensor category. Moreover, the maps given in Equation (2.4) define a pivotal structure on $\mathscr{D}^H$. There is a forgetful functor from $\mathscr{D}^H$ to $\mathscr{D}$ which forgets the action of h_1 and h_2. Given two objects V, W of $\mathscr{D}^H$ let $\mathcal{K} : V \otimes W \to V \otimes W$ be the operator defined by

$$\mathcal{K}(v \otimes w) = q^{-\lambda_1\mu_2 - \lambda_2\mu_1 - 2\lambda_2\mu_2} v \otimes w$$

where $h_i v = \lambda_i v$ and $h_i w = \mu_i w$ for $i = 1, 2$. Consider the truncated R-matrix:

$$\check{R} = \sum_{k=0}^{l'-1} \frac{\{1\}^k}{(k)_q!} E_1^k \otimes F_1^k \sum_{s=0}^{1} \frac{(-\{1\})^s}{(s)_q!} E_3^s \otimes F_3^s \sum_{t=0}^{1} \frac{(-\{1\})^t}{(t)_q!} E_2^t \otimes F_2^t \qquad (2.11)$$

where $E_3 = E_1E_2 - q^{-1}E_2E_1, F_3 = F_2F_1 - qF_1F_2$, $(n)_q = \frac{1-q^n}{1-q}$ and $(n)_q! = (1)_q \cdot \ldots \cdot (n)_q$.

Theorem 2.3.5.4. *The family* $\{c_{V,W} : V \otimes W \to W \otimes V\}_{V,W \in \mathscr{D}^H}$ *defined by*

$$c_{V,W}(v \otimes w) = \tau(\check{R}\mathcal{K}(v \otimes w))$$

is a braiding on $\mathscr{D}^H$ *where* τ *is the super flip map* $\tau(v \otimes w) = (-1)^{|w||v|} w \otimes v$.

Proof. The proof is essentially given by Ha in [36]. As mentioned above, in Theorem 3.6 of [36], Ha proves the theorem for odd ordered roots of unity. Ha's proof works for even ordered roots of unity as well. In particular, before Proposition 3.5 of [36] Ha uses the PBW basis of $U_q(\mathfrak{sl}(2|1))$ to define an algebra $\mathcal{U}^<$. In our case, when defining this algebra one should take powers of E_1 and F_1 from 0 to $l' - 1$ not $l - 1$. Then use this algebra to define the projection $p : U_q(\mathfrak{sl}(2|1)) \to \mathcal{U}^<$ and the element

$$\mathcal{R}^< = p \otimes p(\mathcal{R}_q)$$

where $\mathcal{R}_q$ is the R-matrix defined in [50, 84] for $U_q(\mathfrak{sl}(2|1))$ when q is generic. With these modifications the proofs of Proposition 3.5 and Theorem 3.6 in [36] holds word for word for both the even and odd case. Note that at the end of the proof of Theorem 3.6 in [36] Ha says, "The element $\mathcal{R}^<$ has no pole when q is a root of unity of the order l." This is true in our case because we defined $\mathcal{R}^<$ using p which only allows powers of E_1 or F_1 smaller than $l' - 1$ which is analogous to the definition of $\check{R}$ above. $\square$

For $(n, \alpha) \in \mathbb{N} \times \mathbb{C}$ with $0 \leq n \leq l' - 1$ and $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, one can check directly that there is a $U_q^H(\mathfrak{sl}(2|1))$-module $V^H(n, \alpha)$ with basis $\{w^\alpha_{\rho,\sigma,p} | p \in \{0, ..., n\}; \rho, \sigma \in \{0, 1\}\}$ whose action is given by

$$h_1 \cdot w^\alpha_{\rho,\sigma,p} = (\rho - \sigma + n - 2p) w^\alpha_{\rho,\sigma,p}, \quad h_2 \cdot w^\alpha_{\rho,\sigma,p} = (\alpha + \sigma + p) w^\alpha_{\rho,\sigma,p}, \quad (2.12)$$

and Equations (2.5), (2.6), (2.7), and (2.8) with $\tilde{\alpha}$ replaced with α. Moreover, by definition the operators $q^{h_i} = K_i$ are equal on $V^H(n, \alpha)$.

Lemma 2.3.5.5. *For* $n \in \{0, ..., l' - 1\}$ *and* $\alpha \in \mathbb{C}$ *with* $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, *then the actions of* $E_1^{l'}$ *and* $F_1^{l'}$ *are zero on* $V(n, \tilde{\alpha})$ *and* $V^H(n, \alpha)$.

Proof. We will prove the theorem for $V(n, \tilde{\alpha})$ the proof for $V^H(n, \alpha)$ is essential identical. Let us prove the action of $F_1^{l'}$ is zero on $V(n, \tilde{\alpha})$ the proof that $E_1^{l'}$ act as zero is similar and left to the reader.

It is enough to prove that $F_1^{l'} w_{\rho,\sigma,p} = 0$ where $w_{\rho,\sigma,p}$ is any of the basis vectors given in Theorem 2.3.4.3. Equation (2.6) gives the action of F_1 on

$V(n,\tilde{\alpha})$. In particular, if $\rho \neq 1$ and $\sigma \neq 0$ then $F_1 w_{\rho,\sigma,p} = q^{\sigma-\rho} w_{\rho,\sigma,p+1}$. Therefore, in this case,

$$F_1^{l'} w_{\rho,\sigma,p} = q^{l'(\sigma-\rho)} w_{\rho,\sigma,p+l'} = 0$$

since $w_{\rho,\sigma,i} = 0$ if $i \geq l'$.

Now a direct calculation implies:

$$F_1^k w_{1,0,p} = q^{-k} w_{1,0,p+k} - q^{k-2}\left(\sum_{i=0}^{k-1} q^{-2i}\right) w_{0,1,p+k-1}.$$

When $k = l'$ we see each of these terms is zero, since $w_{\rho,\sigma,i} = 0$ if $i \geq l'$ and $\sum_{i=0}^{l'-1} q^{-2i} = \frac{1-q^{-2l'}}{1-q^{-2}} = 0$.

□

Corollary 2.3.5.6. *For $n \in \{0,...,l'-1\}$ and $\alpha \in \mathbb{C}$ with $\bar{\alpha} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ then the $U_q^H(\mathfrak{sl}(2|1))$-module $V^H(n,\alpha)$ is an object in $\mathscr{D}^H$.*

Lemma 2.3.5.7. *(Commutativity Lemma) Let $n, n' \in \mathbb{N}$ such that $0 \leq n, n' \leq l'-1$ and $\tilde{\alpha}, \tilde{\alpha}' \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha}, \bar{\alpha}' \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Let $\{w_{\rho,\sigma,p}\}$ and $\{w'_{\rho',\sigma',p'}\}$ be the basis given in Theorem 2.3.4.3 for $V(n,\tilde{\alpha})$ and $V(n',\tilde{\alpha}')$, respectively. Choose $\alpha, \alpha' \in \mathbb{C}$ such that $[\alpha] = \tilde{\alpha}$ and $[\alpha'] = \tilde{\alpha}'$ in $\mathbb{C}/l\mathbb{Z}$. Then there exists an isomorphism*

$$\psi_{\alpha,\alpha'} : V(n,\tilde{\alpha}) \otimes V(n',\tilde{\alpha}') \to V(n',\tilde{\alpha}') \otimes V(n,\tilde{\alpha})$$

such that

$$\psi_{\alpha,\alpha'}(w_{0,0,0} \otimes w'_{\rho',\sigma',p'}) = q^{-n(\alpha'+\sigma'+p')-\alpha(\rho'-\sigma'+n'-2p')-2\alpha(\alpha'+\sigma'+p')} w'_{\rho',\sigma',p'} \otimes w_{0,0,0}$$

and

$$\psi_{\alpha,\alpha'}(w_{\rho,\sigma,p} \otimes w'_{0,0,0}) = q^{-(\rho-\sigma+n-2p)\alpha'-(\alpha+\sigma+p)n'-2\alpha'(\alpha+\sigma+p)} w'_{0,0,0} \otimes w_{\rho,\sigma,p} + \sum_i c_i x_i \otimes y_i. \tag{2.13}$$

where each x_i is a basis element in $\{w'_{\rho',\sigma',p'}\}$ not equal to $w'_{0,0,0}$.

Proof. Recall the forgetful functor from $\mathscr{D}^H$ to $\mathscr{D}$. Lemma 2.3.5.5 and Corollary 2.3.5.6 imply that $V^H(n,\alpha)$ maps to $V(n,\tilde{\alpha})$ under this functor. Similarly, $V^H(n',\alpha')$ maps to $V(n',\tilde{\alpha}')$. Now the braiding $c_{V^H(n,\alpha),V^H(n',\alpha')}$ of

Theorem 2.3.5.4 under the forgetful functor gives the desired isomorphism $\psi_{\alpha,\alpha'}$ in $\mathscr{D}$.

We have

$$\psi_{\alpha,\alpha'}(w_{\rho,\sigma,p} \otimes w'_{\rho',\sigma',p'}) = \tau(\check{R}\mathcal{K}(w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{\rho',\sigma',p'}))$$

where

$$\mathcal{K}(w^{\alpha}_{0,0,0} \otimes {w'}^{\alpha'}_{\rho',\sigma',p'}) = q^{-n_1(\alpha'+\sigma'+p')-\alpha(\rho'-\sigma'+n'-2p')-2\alpha(\alpha'+\sigma'+p')} w^{\alpha}_{0,0,0} \otimes {w'}^{\alpha'}_{\rho',\sigma',p'}$$

Since $E_1 w^{\alpha}_{0,0,0} = E_2 w^{\alpha}_{0,0,0} = 0$ it follows that $\check{R}(w^{\alpha}_{0,0,0} \otimes {w'}^{\alpha'}_{\rho',\sigma',p'}) = w^{\alpha}_{0,0,0} \otimes {w'}^{\alpha'}_{\rho',\sigma',p'}$ and the first formula in the lemma holds.

To prove the second formula, recall from Equation (2.11) that

$$\check{R} = 1 \otimes 1 + \sum_i d_i a_i \otimes b_i$$

where each b_i is of the form $F_1^k F_3^s F_2^t$ where at least one of the indices k, s or t is non-zero. Therefore, from the defining relations of Theorem 2.3.4.3 we have $b_i {w'}^{\alpha'}_{0,0,0}$ is a linear combination of basis vectors $w^{\alpha'}_{\rho',\sigma',p'}$ where ρ', σ', p' are not all zero (since the action of either F_1 or F_2 on any basis vector increase at least one of the indices of the vector, see Equation (2.6)). Combining the above we have

$$\begin{aligned}\check{R}(w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{0,0,0}) &= w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{0,0,0} + \sum_i d_i(a_i \otimes b_i)(w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{0,0,0}) \\ &= w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{0,0,0} + \sum_j d'_j y_j \otimes x_j\end{aligned}$$

where each x_i is a basis element in $\{{w'}^{\alpha'}_{\rho',\sigma',p'}\}$ not equal to ${w'}^{\alpha'}_{0,0,0}$. Thus, since $\mathcal{K}$ acts diagonally on the basis, we just need to compute $\mathcal{K}(w^{\alpha}_{\rho,\sigma,p} \otimes {w'}^{\alpha'}_{0,0,0})$. This can be done as above to obtain Equation (2.13). $\square$

Remark 2.3.5.8. *Clearly, the isomorphism $\psi_{\alpha,\alpha'}$ in Lemma 2.3.5.7 depends on the choice of α and α'.*

Lemma 2.3.5.9. *Consider $\tilde{\alpha}_1, ..., \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\alpha}_i \notin \mathcal{Y}, i \in \{1, ..., n\}$. From the Lemma 2.3.5.3, there exists i, j such that $\bar{\alpha}_i + \bar{\alpha}_j \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Then, for any $\epsilon \in \mathbb{C}/l\mathbb{Z}$ such that: $\bar{\alpha}_j - \bar{\epsilon} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ and $\bar{\alpha}_i + \bar{\epsilon} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ we can modify* *the weights without changing the tensor product in the following way:*

$$V(0,\tilde\alpha_1)\otimes ...\otimes V(0,\tilde\alpha_i)\otimes ...\otimes V(0,\tilde\alpha_j)\otimes ...\otimes V(0,\tilde\alpha_n)\simeq$$
$$V(0,\tilde\alpha_1)\otimes ...\otimes V(0,\tilde\alpha_i+\epsilon)\otimes ...\otimes V(0,\tilde\alpha_j-\epsilon)\otimes ...\otimes V(0,\tilde\alpha_n).$$

Proof. From the choice of ϵ, Lemma 2.3.4.6 implies

$$V(0,\tilde\alpha_i)\otimes V(0,\tilde\alpha_j)\simeq V(0,\tilde\alpha_i+\epsilon)\otimes V(0,\tilde\alpha_j-\epsilon).$$

In the following expression, we will use the notation $\hat V$ as a term in a tensor product, in order to say that we skip the module V from this product. Combining this isomorphism with Lemma 2.3.5.7 we have the following isomorphisms:

$$V(0,\tilde\alpha_1)\otimes ...\otimes V(0,\tilde\alpha_i)\otimes ...\otimes V(0,\tilde\alpha_j)\otimes ...\otimes V(0,\tilde\alpha_n)$$
$$\simeq V(0,\tilde\alpha_1)\otimes ...\otimes \hat V(0,\tilde\alpha_i)\otimes ...\otimes \hat V(0,\tilde\alpha_j)\otimes ...\otimes V(0,\tilde\alpha_n)\otimes V(0,\tilde\alpha_i)\otimes V(0,\tilde\alpha_j)$$
$$\simeq V(0,\tilde\alpha_1)\otimes ...\otimes \hat V(0,\tilde\alpha_i)\otimes ...\otimes \hat V(0,\tilde\alpha_j)\otimes ...\otimes V(0,\tilde\alpha_n)\otimes V(0,\tilde\alpha_i+\epsilon)\otimes V(0,\tilde\alpha_j-\epsilon)$$
$$\simeq V(0,\tilde\alpha_1)\otimes ...\otimes V(0,\tilde\alpha_i+\epsilon)\otimes ...\otimes V(0,\tilde\alpha_j-\epsilon)\otimes ...\otimes V(0,\tilde\alpha_n).$$

This concludes the proof. □

2.4 The right trace and its modified dimension

2.4.1 The existence of the right trace

In Subsection 2.2.4 we recalled several results about right traces. Here we apply these results to construct a right trace on the ideal generated by $V(0,\tilde\alpha)$ for $\bar\alpha\notin\{\bar 0,\overline{\frac{l}{2}}\}$.

We've seen in the Decomposition Lemma 2.3.4.5 that for $\tilde\alpha,\tilde\beta\in\mathbb{C}/l\mathbb{Z}$ such that $\bar\alpha,\bar\beta,\bar\alpha+\bar\beta\notin\{\bar 0,\overline{\frac{l}{2}}\}$ we have the following decomposition:

$$V(0,\tilde\alpha)\otimes V(0,\tilde\beta)=V(0,\tilde\alpha+\tilde\beta)\oplus V(0,\tilde\alpha+\tilde\beta+1)\oplus V(1,\tilde\alpha+\tilde\beta).$$

In the case

$$V(0,\tilde\alpha)\otimes V(0,\tilde\alpha)^*=V(0,\tilde\alpha)\otimes V(0,-\tilde\alpha-1)$$

the decomposition is no longer semi-simple, and the two 4-dimensional modules corresponding to $V(0,-1)$ and $V(0,0)$ merge into an indecomposable non-simple 8-dimensional module which we will denote by $V_1(\tilde{\alpha})$. More precisely we have the following result:

Proposition 2.4.1.1. *Let $\tilde{\alpha} \in \mathbb{C}/\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. We have the following decomposition:*

$$V(0,\tilde{\alpha}) \otimes V(0,\tilde{\alpha})^* = V_1(\tilde{\alpha}) \oplus V_2(\tilde{\alpha}) \tag{2.14}$$

where $V_2(\tilde{\alpha})$ is an 8-dimensional simple module and $V_1(\tilde{\alpha})$ is an indecomposable module such that $\mathrm{Hom}_{\mathscr{C}}(\mathbb{C}, V_1(\tilde{\alpha}))$ and $\mathrm{Hom}_{\mathscr{C}}(V_1(\tilde{\alpha}), \mathbb{C})$ are both non-zero.

Proof. Recall $V(0,\tilde{\alpha})^*$ is isomorphic to $V(0,-\tilde{\alpha}-1)$. Let $\{w^{\tilde{\alpha}}_{\rho,\sigma}\}$ and $\{w^{-\tilde{\alpha}-1}_{\rho,\sigma}\}$ be the bases of $V(0,\tilde{\alpha})$ and $V(0,-\tilde{\alpha}-1)$ given in Theorem 2.3.4.3. Consider the vectors of $V(0,\tilde{\alpha}) \otimes V(0,-\tilde{\alpha}-1)$:

$$v_7 = q^{-\tilde{\alpha}-1}[\tilde{\alpha}]w^{\tilde{\alpha}}_{1,1} \otimes w^{-\tilde{\alpha}-1}_{0,0} + q^{\tilde{\alpha}}[\tilde{\alpha}+1]w^{\tilde{\alpha}}_{0,0} \otimes w^{-\tilde{\alpha}-1}_{1,1}$$

and

$$u_0 = [\alpha]w^{\tilde{\alpha}}_{0,0} \otimes w^{-\tilde{\alpha}-1}_{1,0} + q^{-\tilde{\alpha}}[\alpha+1]w^{\tilde{\alpha}}_{1,0} \otimes w^{-\tilde{\alpha}-1}_{0,0}.$$

Let $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ be the modules generated by v_7 and u_0, respectively. The action of these modules is given in Tables 2.1 and 2.2 where $\{v_i\}$ and $\{u_i\}$ are bases for the corresponding modules.

We will show that module $V_1(\tilde{\alpha})$ is indecomposable. Suppose W_1 and W_2 are modules such $V_1(\tilde{\alpha}) = W_1 \oplus W_2$. Since $\{v_i\}$ is a basis of $V_1(\tilde{\alpha})$ there exists

$$v = c_0v_0 + c_1v_1 + c_2v_2 + ... + c_7v_7$$

such that $c_7 \neq 0$ and $v \in W_1$ or $v \in W_2$. Without loss of generality assume $v \in W_1$. From Table 2.1 we have $F_2E_2E_1E_2v$ is a non-zero multiple of v_1. So $v_1 \in W_1$. Then Table 2.1 implies that

$$\{v_1, c^{-1}E_2v_1, F_1v_1, F_2F_1v_1\} = \{v_0, v_1, v_2, v_3\} \subset W_1.$$

Similarly, $E_2F_2F_1F_2v$ is a non-zero multiple of v_5 so $v_5 \in W_1$ and

$$\{v_5, E_1v_5, -c^{-1}F_2v_5\} = \{v_4, v_5, v_6\} \subset W_1.$$

Since W_1 is a submodule we have

$$v_7 = c_7^{-1}(v - c_0v_0 - c_1v_1 - c_2v_2 - ... - c_6v_6) \in W_1$$

Thus, $W_1 = V_1(\tilde{\alpha})$ and we have showed that $V_1(\tilde{\alpha})$ is indecomposable.

Next we will show that $V_2(\tilde{\alpha})$ is simple. Suppose U is a non-zero submodule of $V_2(\tilde{\alpha})$. Notice that the generator u_0 of $V_2(\tilde{\alpha})$ is a highest weight vector. The idea is to push any non-zero vector of U to a multiple of u_0. So let u be a non-zero vector of U. Write u in terms of the basis $\{u_i\}$:

$$u = c_0u_0 + c_1u_1 + c_2u_2 + ... + c_7u_7.$$

If there exists an element x in $U_q(\mathfrak{sl}(2|1))$ such that xu is a non-zero multiple of u_0 then since u_0 is a generator of $V_2(\tilde{\alpha})$ we would have $U \cong V_2(\tilde{\alpha})$. We will show this is true for all possible non-zero coefficients of u.

1. If $c_4 \neq 0$ then from the $U_q(\mathfrak{sl}(2|1))$-action given in the above table we have $E_1E_2E_1E_2u$ is a non-zero multiple of u_0.
2. If $c_4 = 0$ and $c_7 \neq 0$ then $E_2E_1E_2u$ is a non-zero multiple of u_0.
3. If $c_4 = c_7 = 0$ and $c_3 \neq 0$ then $E_1E_2E_1u$ is a non-zero multiple of u_0.
4. If $c_4 = c_7 = c_3 = 0$ and $c_6 \neq 0$ then E_1E_2u is a non-zero multiple of u_0.
5. If $c_4 = c_7 = c_3 = c_6 = 0$ and $c_2 \neq 0$ then E_2E_1u is a non-zero multiple of u_0.
6. If $c_4 = c_7 = c_3 = c_6 = c_2 = 0$ and $c_5 \neq 0$ then E_2u is a non-zero multiple of u_0.
7. Finally, if $c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ and $c_1 \neq 0$ then E_1u is a non-zero multiple of u_0.
8. Finally, if $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$ then $c_0 \neq 0$ and u is a non-zero multiple of u_0.

Thus, $U \cong V_2(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ is simple.

Next, we consider the head and socle of $V_1(\tilde{\alpha})$. We have

$$v_3 = q^{2(-\tilde{\alpha}-1)}w^{\tilde{\alpha}}_{1,1}\otimes w^{-\tilde{\alpha}-1}_{0,0} - q^{-\tilde{\alpha}-1}w^{\tilde{\alpha}}_{0,1}\otimes w^{-\tilde{\alpha}-1}_{1,0} + q^{-\tilde{\alpha}}w^{\tilde{\alpha}}_{1,0}\otimes w^{-\tilde{\alpha}-1}_{0,1} + w^{\tilde{\alpha}}_{0,0}\otimes w^{-\tilde{\alpha}-1}_{1,1}$$

which generates the trivial module in $V_1(\tilde{\alpha})$. Thus, $\mathrm{Hom}_{\mathscr{C}}(\mathbb{C}, V_1(\tilde{\alpha}))$ is non-zero. Also, from Table 2.1 we can see the map

$$V_1(\tilde{\alpha}) \to \mathbb{C} \text{ given by } c_0v_0 + c_1v_1 + ... + c_7v_7 \mapsto c_7$$

is a $U_q(\mathfrak{sl}(2|1))$-module morphism. Thus, $\mathrm{Hom}_{\mathscr{C}}(V_1(\tilde{\alpha}), \mathbb{C})$ is non-zero.

Finally, we prove that Equation (2.14) holds. Since the dimension of $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^*$ is equal to the sum of the dimensions of $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$, it suffices to show that $V_1(\tilde{\alpha}) \cap V_2(\tilde{\alpha}) = \{0\}$. Suppose this is not true. Then there exists a non-zero $v \in V_1(\tilde{\alpha}) \cap V_2(\tilde{\alpha})$. Since $V_2(\tilde{\alpha})$ is simple then $V_2(\tilde{\alpha})$ is isomorphic to the module $< v >$ generated by v. But since $v \in V_1(\tilde{\alpha})$ then $V_2(\tilde{\alpha}) \cong < v > \subset V_1(\tilde{\alpha})$. Since $V_1(\tilde{\alpha})$ and $V_2(\tilde{\alpha})$ have the same dimension this implies that $V_2(\tilde{\alpha}) \cong V_1(\tilde{\alpha})$ which is a contradiction because $V_1(\tilde{\alpha})$ contains the trivial module as a submodule and $V_2(\tilde{\alpha})$ is simple. Thus, we have the decomposition. ☐

Corollary 2.4.1.2. *Let $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. Then $V(0, \tilde{\alpha})$ is a right ambidextrous object in the category $\mathscr{C}$.*

Proof. Equation (2.14) gives a decomposition of $V(0, \tilde{\alpha}) \otimes V(0, \tilde{\alpha})^*$ into indecomposable as in Equation (2.2) where $W_1 = V_1(\tilde{\alpha})$ and $W_2 = V_2(\tilde{\alpha})$. Since $W_2 = V_2(\tilde{\alpha})$ is an 8-dimensional simple module then $\mathrm{Hom}_{\mathscr{C}}(\mathbb{C}, W_2) = \mathrm{Hom}_{\mathscr{C}}(W_2, \mathbb{C}) = 0$. From Lemma 2.2.4.2 there are unique $j, j' \in \{0, 1\}$ such that $\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, W_j)$ and $\mathrm{Hom}_{\mathscr{C}}(W_{j'}, \mathbb{I})$ are non-zero. Thus, $j = j' = 1$ and Theorem 2.2.4.3 implies $V(0, \tilde{\alpha})$ is right ambidextrous. ☐

2.4.2 The modified trace

From Theorem 10 of [27] (for a statement see Theorem 2.2.4.1 above) we have that the right ambidextrous object $V(0, \tilde{\alpha})$ gives a unique right trace:

Theorem 2.4.2.1 ([27])**.** *Let $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. There exists a non-zero right trace $\{\mathsf{t}_V\}$ on the ideal $\mathcal{I}_{V(0,\tilde{\alpha})}$ which is unique up to multiplication by a non-zero scalar.*

The following lemma shows that the ideal generated by $V(0, \tilde{\alpha})$ contains all objects of $\mathscr{C}$ except the trivial module $\mathbb{C}$ and thus the right trace is defined on all these objects. It follows that this ideal is independent of $\tilde{\alpha}$ and we will denote it by $\mathcal{I}$.

Lemma 2.4.2.2. *For any $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ we have $\mathcal{I}_{V(0,\tilde{\alpha})} = \mathscr{C} \setminus \{\mathbb{C}\}$.*

Proof. First, we will show $\mathcal{I}_{V(0,\tilde{\alpha})} \subseteq \mathscr{C} \setminus \{\mathbb{C}\}$. By definition this ideal is contained in $\mathscr{C}$ so we only need to show $\mathbb{C} \notin \mathcal{I}_{V(0,\tilde{\alpha})}$. Suppose on the contrary that $\mathcal{I}_{V(0,\tilde{\alpha})} = \mathscr{C}$. From Lemma 2.2.4.2 and Theorem 2.2.4.3 it follows that the trivial module $\mathbb{C}$ is right ambidextrous. By Theorem 2.2.4.1 there is a unique right trace on $\mathcal{I}_{\mathbb{C}} = \mathscr{C}$. It is easy to see this trace is equal to the usual quantum trace in $\mathscr{C}$ and its associated modified dimension is the usual quantum dimension qdim. Since $\mathcal{I}_{V(0,\tilde{\alpha})} = \mathscr{C} = \mathcal{I}_{\mathbb{C}}$ then from the proof of Lemma 4.2.2 in [26] we have $\mathrm{qdim}(V(0,\tilde{\alpha})) \neq 0$ (note [26] requires a braiding but it is easy to see the cited proof works in our pivotal context). But this is a contradiction so we have the desired inclusion.

Now, we will show the converse inclusion. First, notice that if $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ satisfies $\bar{\beta}, \bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ then Lemma 2.3.4.5 implies that $V(0, \tilde{\alpha} + \tilde{\beta})$ is a retract of $V(0,\tilde{\alpha}) \otimes V(0,\tilde{\beta})$. Therefore, $V(0,\tilde{\alpha}+\tilde{\beta}) \in \mathcal{I}_{V(0,\tilde{\alpha})}$. Now if $\tilde{\mu} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\mu} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ then there exists $\tilde{\beta}, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\beta}, \bar{\gamma} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$, $\tilde{\mu} = \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}$ and $\tilde{\alpha} + \tilde{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. Lemma 2.3.4.5 implies $V(0,\tilde{\mu})$ is a retract of $V(0,\tilde{\alpha}+\tilde{\beta}) \otimes V(0,\tilde{\gamma})$. We have proved that if $\tilde{\mu} \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\mu} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ then $V(0,\tilde{\mu}) \in \mathcal{I}_{V(0,\tilde{\alpha})}$.

Now let $V \in \mathscr{C} \setminus \{\mathbb{C}\}$. By definition of $\mathscr{C}$ there exists $\tilde{\alpha}_1, ..., \tilde{\alpha}_n \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\alpha}_1, ..., \bar{\alpha}_n \notin \mathcal{Y}$ such that V is a retract of $V(0,\tilde{\alpha}_1) \otimes V(0,\tilde{\alpha}_2) \otimes ... \otimes V(0,\tilde{\alpha}_n)$. Since $V(0,\tilde{\alpha}_1)$ is in the ideal $\mathcal{I}_{V(0,\tilde{\alpha})}$ then $V \in \mathcal{I}_{V(0,\tilde{\alpha})}$. □

2.4.3 Computations of modified dimensions

In the proof of Theorem 2.5.2.4 we will show that the ideal $\mathcal{I}$ contains $V(n,\tilde{\alpha})$ such that $0 \leq n \leq l' - 2$ and $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. In the next lemma we will compute the modified quantum dimension of such modules. Recall that the modified quantum dimension is defined to be $\mathsf{d}(V) = \mathsf{t}_V(\mathrm{Id}_V)$ for $V \in \mathcal{I}$.

Lemma 2.4.3.1. *If $V(n,\tilde{\alpha}) \in \mathcal{I}$ for $0 \leq n \leq l' - 1$ and $\bar{\alpha} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ then the right trace $\{\mathsf{t}_V\}_{V \in \mathcal{I}}$ can be normalized so*

$$\mathsf{d}(V(n,\tilde{\alpha})) = \frac{\{n+1\}}{\{1\}\{\tilde{\alpha}\}\{\tilde{\alpha}+n+1\}} \tag{2.15}$$

where $\{z\} = q^z - q^{-z}$.

Proof. In the proof of Lemma 2.4.2.2 we showed that $V(0, \frac{\tilde{1}}{3}) \in \mathcal{I}$. The trace is unique up to a global scalar and we choose a normalization so that

$$\mathsf{d}(V(0, \frac{\tilde{1}}{3})) = \mathsf{t}_{V(0,\frac{\tilde{1}}{3})}(\mathrm{Id}_{V(0,\frac{\tilde{1}}{3})}) = \frac{1}{\{\frac{\tilde{1}}{3}\}\{\frac{\tilde{1}}{3}+1\}}.$$

Let $V(n, \tilde{\alpha}) \in \mathcal{I}$ for $0 \leq n \leq l' - 1$ and $\bar{\alpha} \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Fix $\alpha \in \mathbb{C}$ such that $\tilde{\alpha} = \alpha$ modulo $l\mathbb{Z}$. For $U, W \in \mathscr{C}$ and $f \in \mathrm{End}_{\mathscr{C}}(U \otimes W)$ define

$$\mathrm{ptr}^W(f) = \Big((\mathrm{Id}_U \otimes \overleftarrow{\mathrm{ev}}_W)(f \otimes \mathrm{Id}_{W^*})(\mathrm{Id}_U \otimes \overrightarrow{\mathrm{coev}}_W)\Big).$$

Recall the isomorphisms $\psi_{\alpha,\frac{1}{3}}$ and $\psi_{\frac{1}{3},\alpha}$ of Lemma 2.3.5.7.

Let $S'_{\alpha,\frac{1}{3}}$ and $S'_{\frac{1}{3},\alpha}$ be the complex numbers defined by the following equations:

$$S'_{\frac{1}{3},\alpha}\,\mathrm{Id}_{V(n,\tilde{\alpha})} = \mathrm{ptr}^{V(0,\frac{\tilde{1}}{3})}\left(\psi_{\frac{1}{3},\alpha}\psi_{\alpha,\frac{1}{3}}\right), \quad S'_{\alpha,\frac{1}{3}}\,\mathrm{Id}_{V(0,\frac{\tilde{1}}{3})} = \mathrm{ptr}^{V(n,\tilde{\alpha})}\left(\psi_{\alpha,\frac{1}{3}}\psi_{\frac{1}{3},\alpha}\right).$$

Now from properties of the modified trace we have

$$\begin{aligned}
\mathsf{d}(V(n,\tilde{\alpha}))S'_{\frac{1}{3},\alpha} &= \mathsf{t}_{V(n,\tilde{\alpha})}\left(\mathrm{ptr}^{V(0,\frac{\tilde{1}}{3})}\left(\psi_{\frac{1}{3},\alpha}\psi_{\alpha,\frac{1}{3}}\right)\right)\\
&= \mathsf{t}_{V(n,\tilde{\alpha})\otimes V(0,\frac{\tilde{1}}{3})}\left(\psi_{\frac{1}{3},\alpha}\psi_{\alpha,\frac{1}{3}}\right)\\
&= \mathsf{t}_{V(0,\frac{\tilde{1}}{3})\otimes V(n,\tilde{\alpha})}\left(\psi_{\alpha,\frac{1}{3}}\psi_{\frac{1}{3},\alpha}\right)\\
&= \mathsf{t}_{V(0,\frac{\tilde{1}}{3})}\left(\mathrm{ptr}^{V(n,\tilde{\alpha})}\left(\psi_{\alpha,\frac{1}{3}}\psi_{\frac{1}{3},\alpha}\right)\right)\\
&= \mathsf{d}\left(V\big(0,\frac{\tilde{1}}{3}\big)\right)S'_{\alpha,\frac{1}{3}}.
\end{aligned}$$

Now, if $S'_{\frac{1}{3},\alpha} \neq 0$ (which we will show below) then

$$\mathsf{d}(V(n,\tilde{\alpha})) = \frac{S'_{\alpha,\frac{1}{3}}}{\left(\{\frac{\tilde{1}}{3}\}\{\frac{\tilde{1}}{3}+1\}\right)S'_{\frac{1}{3},\alpha}}. \tag{2.16}$$

Thus, to find a formula for $\mathsf{d}(V(n,\tilde{\alpha}))$ it suffices to compute $S'_{\frac{1}{3},\alpha}$ and $S'_{\alpha,\frac{1}{3}}$.

We now compute $S'_{\frac{1}{3},\alpha}$. Let $\{w_{\rho,\sigma,p}\}_{\rho,\sigma\in\{0,1\},p\in\{0,\dots n-1\}}$ and $\{w'_{\rho',\sigma',0}\}_{\rho',\sigma'\in\{0,1\}}$ is the weight bases of $V(n,\tilde{\alpha})$ and $V(0,\frac{\tilde{1}}{3})$, respectively. Any endomorphism of $V(n,\tilde{\alpha})$ maps the highest weight vector $w_{0,0,0}$ of $V(n,\tilde{\alpha})$ to a multiple of itself. Since $V(n,\tilde{\alpha})$ is simple it is enough to compute this coefficient, in other words

$$\mathrm{ptr}^{V(0,\tilde{\frac{1}{3}})}\left(\psi_{\frac{1}{3},\alpha}\psi_{\alpha,\frac{1}{3}}\right)(w_{0,0,0}) = S'_{\frac{1}{3},\alpha}w_{0,0,0}. \tag{2.17}$$

Now $\psi_{\frac{1}{3},\alpha}$ and $\psi_{\alpha,\frac{1}{3}}$ are determined by the action of the R-matrix $\check{R}\mathcal{K}$ on the $U_q^H(\mathfrak{sl}(2|1))$-modules $V^H(n,\alpha)$ and $V^H(0,\frac{1}{3})$. Since we are taking a partial trace only diagonal quantities of this action contribute when writing on the weight vector basis $\{w^{\frac{1}{3}}_{\rho,\sigma,0}\}_{\rho,\sigma\in\{0,1\}}$ of $V^H(0,\frac{1}{3})$ given above. So it is enough to know the values of $\psi_{\alpha,\frac{1}{3}}(w_{0,0,0}\otimes w'_{\rho',\sigma',0})$ and $\psi_{\frac{1}{3},\alpha}(w'_{\rho',\sigma',0}\otimes w_{0,0,0})$ which are computed in Lemma 2.3.5.7. Note, the terms $c_ix_i\otimes y_i$ in Equation (2.13) are not diagonal and so do not contribute. Thus, evaluating the left side of Equation (2.17) we have

$$\begin{aligned}
S'_{\frac{1}{3},\alpha} &= \sum_{\rho',\sigma'=0}^{1} q^{(-2n-4\alpha)(\frac{1}{3}+\sigma')-2\alpha(\rho'-\sigma')}(-1)^{\rho'+\sigma'}w'^{*}_{\rho',\sigma',0}\left(K_2^{-2}w'_{\rho',\sigma',0}\right)\\
&= \sum_{\rho',\sigma'=0}^{1} q^{(-2n-4\alpha)(\frac{1}{3}+\sigma')-2\alpha(\rho'-\sigma')-2(\tilde{\frac{1}{3}}+\sigma')}(-1)^{\rho'+\sigma'}\\
&= q^{(-\frac{2}{3}-1)(2\alpha+n+1)}\left(q^{2\alpha+n+1}-q^{-n-1}-q^{n+1}+q^{-2\alpha-n-1}\right)\\
&= q^{-(\frac{2}{3}+1)(2\alpha+n+1)}\{\alpha\}\{\alpha+n+1\}.
\end{aligned}$$

Similarly,

$$S'_{\alpha,\frac{1}{3}} = q^{-(2\alpha+n+1)(\frac{2}{3}+1)}\frac{\{n+1\}}{\{1\}}\{1/3\}\{4/3\}.$$

Finally, since $\{\tilde{x}\}=\{x\}$ for any $x\in\mathbb{C}$ then Equation (2.16) implies the result. $\square$

2.5 The relative $\mathbb{C}/\mathbb{Z}$-spherical category

2.5.1 Purification of $\mathscr{C}$

The category $\mathscr{C}$ that we've constructed still needs to be modified in order to obtain a relative G-spherical category. One of the main problem is that there are an infinite number of non-isomorphic simple objects in each graded piece of $\mathscr{C}$. In order to obtain a finite number of objects in each grading, we will "purify" the category using the modified trace. This will have the effect of removing all modules outside the alcove. This generalizes the well known purification of a category discussed in Chapter XI of [81].

Let $V, W \in \mathcal{I} = \mathscr{C} \setminus \{\mathbb{C}\}$. A morphism $f \in \mathrm{Hom}_{\mathscr{C}}(V, W)$ is called negligible with respect to the right trace t if

$$\mathsf{t}_W(f \circ g) = \mathsf{t}_V(g \circ f) = 0$$

for all $g \in \mathrm{Hom}_{\mathscr{C}}(W, V)$. Denote $\mathrm{Negl}(V, W)$ as the set of negligible morphisms from V to W. The set $\mathrm{Negl}(V, W)$ is actually a sub-vector space of $\mathrm{Hom}_{\mathscr{C}}(W, V)$. Thus, we can take the quotient and obtain a $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathscr{C}}(V, W)/\mathrm{Negl}(V, W)$. We set $\mathrm{Negl}(V, \mathbb{C}) = \mathrm{Negl}(\mathbb{C}, V) = 0$ for any $V \in \mathscr{C}$.

We describe a purification process of $\mathscr{C}$ which will produce a category $\mathscr{C}^N$ where all negligible morphism are zero. We define a new pivotal $\mathbb{C}$-category $\mathscr{C}^N$ whose objects are the same as in $\mathscr{C}$. The set of morphisms between two objects V and W of $\mathscr{C}^N$ is

$$\mathrm{Hom}_{\mathscr{C}^N}(V, W) = \mathrm{Hom}_{\mathscr{C}}(V, W)/\mathrm{Negl}(V, W).$$

The composition, tensor product, pivotal structure and grading in $\mathscr{C}^N$ is induced from $\mathscr{C}$:

Lemma 2.5.1.1. *The category $\mathscr{C}^N$ is a pivotal $\mathbb{C}$-category with a $\mathbb{C}/\mathbb{Z}$-grading induced from the grading of $\mathscr{C}$.*

Proof. First, we will show that $\mathscr{C}^N$ is a pivotal $\mathbb{C}$-category. There is an obvious functor $\mathcal{F} : \mathscr{C} \to \mathscr{C}^N$ which is the identity on objects and maps a morphism to its class modulo negligible morphisms:

1. $\mathcal{F}(A) = A, \forall A \in Ob(\mathscr{C})$,
2. $\mathcal{F}(f) = [f] \in \mathrm{Hom}_{\mathscr{C}^N}(A, B), \forall f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.

Using this functor we can induce the tensor $\mathbb{C}$-linear structure of $\mathscr{C}$ onto $\mathscr{C}^N$. We also define the dual structure on $\mathscr{C}^N$ as the one coming from $\mathscr{C}$, via the functor $\mathcal{F}$. Since the dualities morphisms in $\mathscr{C}$ satisfy the compatibility conditions for a pivotal structure, then the corresponding dualities under $\mathcal{F}$ will also satisfy these compatibility conditions in $\mathscr{C}^N$.

Recall the definition of a G-graded category given in Subsection 2.2.3. From Lemma 2.3.5.2, we know that $\mathscr{C}$ is $\mathbb{C}/\mathbb{Z}$-graded. For any $g \in \mathbb{C}/\mathbb{Z}$, define $\mathscr{C}^N_g := \mathcal{F}(\mathscr{C}_g)$. It is easy to see this gives a $\mathbb{C}/\mathbb{Z}$-graded on $\mathscr{C}^N$. $\square$

We will use the same notation for the object $V(n, \tilde{\alpha})$ of $\mathscr{C}$ and the corresponding object in $\mathscr{C}^N$.

Lemma 2.5.1.2. *If $W \in \mathscr{C}$ such that W is simple and $\mathsf{d}(W) = 0$ then every morphism to or from W is negligible.*

Proof. It suffices to prove that $\mathsf{t}_W(h) = 0$ for any $h \in \mathrm{End}_{\mathscr{C}}(W)$. This will imply that if $V \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(V, W)$ then

$$\mathsf{t}_W(f \circ g) = 0$$

for any $g \in \mathrm{End}_{\mathscr{C}}(W, V)$. Thus, f is negligible. A similar statement holds for $f \in \mathrm{Hom}_{\mathscr{C}}(W, V)$.

To prove the first statement, let $h \in \mathrm{End}_{\mathscr{C}}(W)$. Since W is simple, $\mathrm{End}_{\mathscr{C}}(W) = \mathbb{C}\,\mathrm{Id}_W$ and we will define the scalar $< h > \in \mathbb{C}$ as the solution to the equation $h =< h > \mathrm{Id}_W$. But $\mathsf{d}(W) = 0$, in other words $\mathsf{t}_W(\mathrm{Id}_W) = 0$. Thus,

$$\mathsf{t}_W(h) = \mathsf{t}_W(< h > \mathrm{Id}_W) =< h > \mathsf{t}_W(\mathrm{Id}_W) = 0.$$

□

Lemma 2.5.1.3. *Let V and W be objects in $\mathscr{C}$ such that W is simple and $\mathsf{d}(W) = 0$. Then V, W are also object in $\mathscr{C}^N$ with the property that the direct sum $V \oplus W$ is isomorphic to V in $\mathscr{C}^N$, in other words $V \oplus W \simeq_{\mathscr{C}^N} V$.*

Proof. Let $i_1 : V \to V \oplus W$ and $pr_1 : V \oplus W \to V$ be the injection and projection morphisms with $pr_1 \circ i_1 = \mathrm{Id}_V$ in $\mathscr{C}$. This gives the relation $pr_1 \circ i_1 = \mathrm{Id}_V$ in $\mathscr{C}^N$. We want to show $i_1 \circ pr_1 = \mathrm{Id}_{V \oplus W}$ in $\mathscr{C}^N$. To do this consider the other inclusion and projection morphisms $i_2 : W \to V \oplus W$ and $pr_2 : V \oplus W \to W$ in $\mathscr{C}$. Then by definition $\mathrm{Id}_{V\oplus W} = i_1 \circ pr_1 + i_2 \circ pr_2$ in $\mathscr{C}$. But from Lemma 2.5.1.2 we have $i_2 \circ pr_2$ is negligible. Thus, in $\mathscr{C}^N$ we have $\mathrm{Id}_{V\oplus W} = i_1 \circ pr_1$ and so i_1 is the inverse of pr_1. □

Corollary 2.5.1.4. *Let $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\gamma} \notin \{\bar{0}, \overline{\tfrac{l}{2}}\}$. If $V(l' - 1, \tilde{\gamma}) \in \mathcal{I}$ then for any $V \in \mathscr{C}^N$ we have $V \oplus V(l' - 1, \tilde{\gamma}) \simeq_{\mathscr{C}^N} V$. In particular, $V(l' - 1, \tilde{\gamma}) \simeq_{\mathscr{C}^N} \{0\}$.*

Proof. From Lemma 2.4.3.1, we have that $\mathsf{d}(V(l' - 1, \tilde{\gamma})) = 0$. Applying the previous lemma we conclude the isomorphism. □

2.5.2 Generically finitely semi-simple

Lemma 2.5.2.1. *Let V be a simple object in $\mathscr{C}$. As an object of $\mathscr{C}^N$, V is either simple or $V \simeq_{\mathscr{C}^N} \{0\}$.*

Proof. Since V is simple we have that $\mathrm{End}_{\mathscr{C}}(V) \simeq \mathbb{C}\cdot\mathrm{Id}_V$ is the 1-dimensional vector space. By definition

$$\mathrm{End}_{\mathscr{C}^N}(V) = \mathrm{End}_{\mathscr{C}}(V)/\operatorname{Negl}(V,V) = (\mathbb{C}\cdot\mathrm{Id}_V)/\operatorname{Negl}(V,V).$$

Thus, $\mathrm{End}_{\mathscr{C}^N}(V)$ is either 0 or 1-dimensional corresponding to the two cases of the lemma. □

Lemma 2.5.2.2. *Let $U \in \mathscr{C}$ such that $U = ({}_{\mathscr{C}} \oplus_{j\in J} S_j) \oplus W$ where J is a finite indexing set and S_j is simple for all $j \in J$. Let $V \in \mathscr{C}$ be a retract of U with maps $i : V \to U$ and $p : U \to V$. Then the following statements are true:*

1) There exist $J' \subseteq J$ and $W' \subseteq W$ such that:

$$Im(i) = (\oplus_{j\in J'} S_j) \oplus W'.$$

Moreover, if $i' : V \to Im(i)$ is the function i but with range $Im(i)$ then i' is an isomorphism with inverse $p' := p|_{Im(i)}$.

2) W' is a retract of W.

Proof. 1) Denote by $p_j : U \to S_j$ and $p_W : U \to W$ the projections onto direct summands of U.

Consider $J' := \{j \in J | p_j \circ i \neq 0\}$ and $W' = Im(p_W \circ i)$. Since for any $j \in J$, S_j is simple then it is generated by any non-zero element. Using this and the fact that $p_j \circ i : V \to S_j$ is a non-zero morphism for all $j \in J'$, we obtain that this morphism is surjective. We conclude that $Im(i) =_{\mathscr{C}} (\oplus_{j\in J'} S_j) \oplus W'$. So i' is surjective and injective. Moreover, $p' \circ i' = p \circ i = \mathrm{Id}_V$.

2) We prove the second statement in two steps.

Step 1. We will show that $(\oplus_{j\in J'} S_j) \oplus W'$ is a retract of $(\oplus_{j\in J} S_j) \oplus W$. Consider $\iota : (\oplus_{j\in J'} S_j) \oplus W' \to (\oplus_{j\in J} S_j) \oplus W$ the natural inclusion, of each component of the direct sum in the left to the corresponding one on the right hand side. From the first part of the proof we have

$$p \circ \iota = p'.$$

Define $\pi : (\oplus_{j\in J} S_j) \oplus W \to (\oplus_{j\in J'} S_j) \oplus W'$ by $\pi := i' \circ p$. Then since i' and p' are inverses we have:

$$\pi \circ \iota = \ (i' \circ p) \circ \iota = \ i' \circ (p \circ \iota) = i' \circ p' = \mathrm{Id}_{(\oplus_{j\in J'} S_j)\oplus W'}\,.$$

This concludes the Step 1.

Second 2. Consider $\iota_{W'} : W' \to (\oplus_{j\in J'} S_j) \oplus W'$ and $\pi_{W'} : (\oplus_{j\in J'} S_j) \oplus W' \to W'$ the injection and projection with respect to the direct summand of W'. Similarly, consider the injection $\iota_W : W \to (\oplus_{j\in J} S_j) \oplus W$ and projection $\pi_W : (\oplus_{j\in J} S_j) \oplus W \to W$.

Define $\iota' : W' \to W$ and $\pi' : W \to W'$ as:

$$\iota' := \pi_W \circ \iota \circ \iota_{W'} \quad \textit{and} \quad \pi' := \pi_{W'} \circ \pi \circ \iota_W.$$

By definition we have:

$$\pi' \circ \iota' = \pi_{W'} \circ \pi \circ (\iota_W \circ \pi_W) \circ \iota \circ \iota_{W'}.$$

Since $Im(\iota \circ \iota_{W'}) \subseteq 0 \oplus W \subseteq (\oplus_{j\in J} S_j) \oplus W$, this means that

$$(\iota_W \circ \pi_W) \circ \iota \circ \iota_{W'} = \iota \circ \iota_{W'}.$$

So, we obtain:

$$\pi' \circ \iota' = \pi_{W'} \circ \pi \circ \iota \circ \iota_{W'}.$$

Using the conclusion of the first step ($\pi \circ \iota = \mathrm{Id}$) we have:

$$\pi' \circ \iota' = \pi_{W'} \circ \iota_{W'} = \mathrm{Id}_{W'}\,.$$

This finishes the proof of the second part. □

Lemma 2.5.2.3. *Let $V, W \in \mathscr{C}$ such that W is retract of V in $\mathscr{C}$. If V is isomorphic to the zero module in $\mathscr{C}^N$ (i.e. $V \simeq_{\mathscr{C}^N} \{0\}$) then so is W:*

$$W \simeq_{\mathscr{C}^N} \{0\}.$$

Proof. Let $i : W \to V$ and $\pi : V \to W$ be the retract in $\mathscr{C}$. Let $[i] : W \to V$ and $[\pi] : V \to W$ be there images in $\mathscr{C}^N$. Since $V \simeq_{\mathscr{C}^N} \{0\}$, this means the zero maps in $\mathscr{C}^N$:

$$[0]_V : V \to \{0\} \quad \textit{and} \quad [0]^V : \{0\} \to V$$

are inverses of each other in $\mathscr{C}^N$. In particular, we have:

$$[0]^V \circ [0]_V =_{\mathscr{C}^N} [\mathrm{Id}_V]. \tag{2.18}$$

Consider the zero maps:

$$[0]_W : W \to \{0\} \quad and \quad [0]^W : \{0\} \to W.$$

We have $[0]_W \circ [0]^W =_{\mathscr{C}^N} [0] =_{\mathscr{C}^N} [\mathrm{Id}_{\{0\}}]$. For the other composition, notice that

$$[0]_W =_{\mathscr{C}^N} [0]_V \circ [i] \quad and \quad [0]^W =_{\mathscr{C}^N} [\pi] \circ [0]^V$$

so we have:

$$[0]^W \circ [0]_W =_{\mathscr{C}^N} ([\pi] \circ [0]^V) \circ ([0]_V \circ [i]) =_{\mathscr{C}^N} [\pi] \circ ([0]^V \circ [0]_V) \circ [i]$$

$$=_{\mathscr{C}^N} \pi \circ [\mathrm{Id}_V] \circ i =_{\mathscr{C}^N} \pi \circ i =_{\mathscr{C}^N} [\mathrm{Id}_W]$$

where third equality comes from Equation (2.18). Thus, we have shown $[0]_W$ and $[0]^W$ are inverses of each other. $\square$

Theorem 2.5.2.4. *Let $g \in G, g \notin \{\bar{0}, \frac{\bar{l}}{2}\}$.*
1) The category $\mathscr{C}_g^N$ is semi-simple.
2) The set of isomorphism classes of simple objects in $\bigcup_{g \in G \setminus \{\bar{0}, \frac{\bar{l}}{2}\}} \mathscr{C}_g^N$ is

$$\left\{ V(n, \tilde{\gamma}) | 0 \leq n \leq l' - 2,\ \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z},\ \bar{\gamma} \notin \left\{ \bar{0}, \frac{\bar{l}}{2} \right\} \right\}.$$

Proof. **Proof of part 1).** To prove the first statement we begin by showing that elementary tensor products of $V(0, \tilde{\alpha})$ which arrive in grading g are semi-simple in $\mathscr{C}_g^N$. To do this we first work in $\mathscr{C}$ and use induction on the number of terms in the tensor product. Then we show such a tensor product is semi-simple in $\mathscr{C}_g^N$.

Let $P(n)$ be the following statement:

> If $h \in G$ and $\tilde{\alpha}_i \in \mathbb{C}/l\mathbb{Z}$ such that $h \notin \{\bar{0}, \frac{\bar{l}}{2}\}$, $\bar{\alpha}_i \notin \mathcal{Y}$ for $i = 1, ..., n$ and $\bar{\alpha}_1 + \bar{\alpha}_2 + \cdots + \bar{\alpha}_n = h$ then as an object in $\mathscr{C}$ the tensor product
>
> $$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) \otimes ... \otimes V(0, \tilde{\alpha}_n)$$
>
> can be written as a direct sum of modules of the following form:

(a) $V(m, \tilde{\beta})$ where $m \leq min\{n-1, l'-2\}$ and $\bar{\beta} = h$,

(b) $V(l'-1, \tilde{\delta}) \otimes W$ where $\bar{\delta} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$ and W is an object of $\mathscr{C}$.

Moreover, this decomposition contains at least one module of the form $V(min\{n-1, l'-2\}, \tilde{\beta})$ for some $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\beta} = h$ and $W = \{0\}$ if $n < l'-1$.

We prove this statement by induction.

The case $n = 2$. Let $h \in G \setminus \{\bar{0}, \bar{\frac{l}{2}}\}$ and $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha}_1 + \bar{\alpha}_2 = h$. It follows from Lemma 2.3.4.5 that:

$$V(0, \tilde{\alpha}_1) \otimes V(0, \tilde{\alpha}_2) =_{\mathscr{C}} V(0, \tilde{\alpha}_1 + \tilde{\alpha}_2) \oplus V(1, \tilde{\alpha}_1 + \tilde{\alpha}_2) \oplus V(0, \tilde{\alpha}_1 + \tilde{\alpha}_2 + 1).$$

As we can see, all the modules have the right form $V(n, \tilde{\alpha})$ with $n \leq 1$ and there is one $V(1, \tilde{\alpha}_1 + \tilde{\alpha}_2)$ which occurs.

Next we assume $P(n)$ **is true and show** $P(n+1)$ **holds.** To do this we need to consider two cases $n \geq l'-1$ and $n < l'-1$.

Case 1: $n \geq l'-1$. Let $\tilde{\alpha}_1, ..., \tilde{\alpha}_{n+1}$ be as in the statement of $P(n+1)$. From Lemma 2.3.5.3, there exists $i, j \in \{1, ..., n+1\}$ such that $\bar{\alpha}_i + \bar{\alpha}_j \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Choose $\epsilon \in \mathbb{C}/l\mathbb{Z}$ such that:

1. $\bar{\alpha}_1 + \bar{\alpha}_2 + ... + \widehat{\bar{\alpha}}_j + ... + \bar{\alpha}_{n+1} + \bar{\epsilon} \neq \bar{0}, \bar{\frac{l}{2}}$,
2. $\bar{\alpha}_j - \bar{\epsilon} \neq \bar{0}, \bar{\frac{l}{2}}$,
3. $\bar{\alpha}_i + \bar{\epsilon} \neq \bar{0}, \bar{\frac{l}{2}}$.

Using Lemma 2.3.5.9 and the Commutativity Lemma 2.3.5.7, we obtain that:

$$\begin{aligned}
&V(0, \tilde{\alpha}_1) \otimes ... \otimes V(0, \tilde{\alpha}_i) \otimes ... \otimes V(0, \tilde{\alpha}_j) \otimes ... \otimes V(0, \tilde{\alpha}_{n+1}) \simeq_{\mathscr{C}} \\
&\quad V(0, \tilde{\alpha}_1) \otimes ... \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes ... \otimes V(0, \tilde{\alpha}_j - \epsilon) \otimes ... \otimes V(0, \tilde{\alpha}_{n+1}) \simeq_{\mathscr{C}} \\
&\quad V(0, \tilde{\alpha}_1) \otimes ... \otimes \widehat{V(0, \tilde{\alpha}_i)} \otimes ... \otimes \widehat{V(0, \tilde{\alpha}_j)} \otimes ... \otimes V(0, \tilde{\alpha}_{n+1}) \otimes V(0, \tilde{\alpha}_i + \epsilon) \otimes V(0, \tilde{\alpha}_j - \epsilon).
\end{aligned}$$

This shows it suffices to prove that the statement $P(n+1)$ holds for weights of the form

$$(\tilde{\alpha}_1, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \tilde{\alpha}_{n+1}, \tilde{\alpha}_i + \epsilon, \tilde{\alpha}_j - \epsilon).$$

By the choice of ϵ we have $\tilde{\alpha}_1, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \tilde{\alpha}_{n+1}, \tilde{\alpha}_i + \epsilon$ has the total grading different than $\bar{0}, \bar{\frac{l}{2}}$, so it satisfies the property in the statement of $P(n)$. Therefore, by the induction hypothesis there exists:

$m_1, ..., m_k \in \{0, ..., l'-3\}, \tilde{\beta}_1, ..., \tilde{\beta}_k, \tilde{\gamma}_1, ..., \tilde{\gamma}_s, \tilde{\delta}_1, ..., \tilde{\delta}_p \in \mathbb{C}/l\mathbb{Z},$ and $W_1, ..., W_p \in \mathscr{C}$

such that

$$V(0,\tilde{\alpha}_1)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_i)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_j)\otimes ...\otimes V(0,\tilde{\alpha}_{n+1})\otimes V(0,\tilde{\alpha}_i+\epsilon)$$
$$\cong_{\mathscr{C}} (\oplus_u V(m_u,\tilde{\beta}_u))\oplus(\oplus_t V(l'-2,\tilde{\gamma}_t))\oplus(\oplus_k(V(l'-1,\tilde{\delta}_k)\otimes W_k))$$

Taking the tensor product with $V(0,\tilde{\alpha}_j-\epsilon)$ we obtain:

$$V(0,\tilde{\alpha}_1)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_i)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_j)\otimes ...\otimes V(0,\tilde{\alpha}_{n+1})\otimes V(0,\tilde{\alpha}_i+\epsilon)\otimes V(0,\tilde{\alpha}_j-\epsilon)$$
$$\cong_{\mathscr{C}} (\oplus_u(V(m_u,\tilde{\beta}_u)\otimes V(0,\tilde{\alpha}_j-\epsilon)))\oplus(\oplus_t(V(l'-2,\tilde{\gamma}_t)\otimes V(0,\tilde{\alpha}_j-\epsilon)))$$
$$\oplus(\oplus_k((V(l'-1,\tilde{\delta}_k)\otimes W_k)\otimes V(0,\tilde{\alpha}_j-\epsilon))).$$

Since the tensor product preserves the grading we have

$$\bar{\beta}_u=\bar{\gamma}_t=\bar{\alpha}_1+...+\hat{\alpha}_j+\bar{\alpha}_{n+1}+\bar{\epsilon}$$

for all $u\in\{1,...,k\}$ and $t\in\{1,...,s\}$. It follows that

$$\bar{\beta}_u+\bar{\alpha}_j-\bar{\epsilon}=\bar{\alpha}_1+...+\bar{\alpha}_{n+1}\notin\{\bar{0},\frac{\bar{l}}{2}\}.$$

Similarly, $\bar{\gamma}_t+\bar{\alpha}_j-\bar{\epsilon}\notin\{\bar{0},\frac{\bar{l}}{2}\}$. Lemma 2.3.4.5 implies that the final expression in the previous tensor decomposition is isomorphic to

$$\oplus_u\big(\ V(m_u,\tilde{\beta}_u+\tilde{\alpha}_j-\epsilon)\oplus V(m_u+1,\tilde{\beta}_u+\tilde{\alpha}_j-\epsilon)\oplus$$
$$\oplus V(m_u-1,\tilde{\beta}_u+\tilde{\alpha}_j-\epsilon+1)\oplus V(m_u,\tilde{\beta}_u+\tilde{\alpha}_j-\epsilon+1)\ \big)\oplus$$
$$\oplus\oplus_t\big(\ V(l'-2,\tilde{\gamma}_t+\tilde{\alpha}_j-\epsilon)\oplus(V(l'-1,\tilde{\gamma}_t+\tilde{\alpha}_j-\epsilon)\otimes\mathbb{I})\oplus$$
$$\oplus V(l'-3,\tilde{\gamma}_t+\tilde{\alpha}_j-\epsilon+1)\oplus V(l'-2,\tilde{\gamma}_t+\tilde{\alpha}_j-\epsilon+1)\ \big)\oplus$$
$$\oplus\oplus_k\big(\ V(l'-1,\tilde{\delta}_k)\otimes W'_k\ \big)$$

where $W'_k=W_k\otimes V(0,\tilde{\alpha}_j-\epsilon)$. We notice that from the induction hypothesis $\bar{\delta}_k\notin\{\bar{0},\frac{\bar{l}}{2}\}$ and from the previous relation $\bar{\gamma}_t+\bar{\alpha}_j-\bar{\epsilon}\notin\{\bar{0},\frac{\bar{l}}{2}\}$, so all the second components that occur are not in $\{\bar{0},\frac{\bar{l}}{2}\}$. Also, notice that the decomposition contains $V(l'-2,\tilde{\gamma}_t+\tilde{\alpha}_j-\epsilon+1)$ as a summand. Thus, we proved the step $P(n+1)$ in this case.

Case 2: $n<l'-1$**.** The proof of the previous case also works here except that things are slightly simpler in this case because no module of the form $V(l'-1,\tilde{\gamma})\otimes W$ appears in the large tensor product. We highlight

the differences: the first part of the proof is the same. Then the induction hypothesis implies there exists:

$$m_1, ..., m_k \in \{0, ..., n-1\}, \text{ and } \tilde{\beta}_1, ..., \tilde{\beta}_k \in \mathbb{C}/l\mathbb{Z}$$

such that

$$V(0,\tilde{\alpha}_1)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_i)\otimes ...\otimes \hat{V}(0,\tilde{\alpha}_j)\otimes ...\otimes V(0,\tilde{\alpha}_{n+1})\otimes V(0,\tilde{\alpha}_i+\epsilon) \cong \oplus_u V(m_u,\tilde{\beta}_u)$$

where at least one $m_i = n-1$. As above take the tensor product with $V(0,\tilde{\alpha}_j - \epsilon)$ then the Decomposition Lemma 2.3.4.5 implies

$$V(m_u,\tilde{\beta}_u) \otimes V(0,\tilde{\alpha}_j - \epsilon) \tag{2.19}$$

decomposes into a direct sum of modules of the form $V(m,\tilde{\beta})$ where $m \leq m_u + 1 \leq n < l' - 1$ and $\bar{\beta} = \bar{\beta}_u + \bar{\alpha}_j - \bar{\epsilon} \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Also notice that when $m_u = m_i = n-1$ then the tensor product in Equation (2.19) as a summand for the form $V(m_i+1,\tilde{\beta}) = V(n,\tilde{\beta})$. Thus, we have proved that the statement for $P(n+1)$ holds.

Now we will show that $\mathscr{C}_g^N$ **is semi-simple**

Let $V \in \mathscr{C}_g^N$. Then, from the definition, V is a $\mathscr{C}$-retract of a module

$$V(0,\tilde{\alpha}_1) \otimes V(0,\tilde{\alpha}_2) \otimes ... \otimes V(0,\tilde{\alpha}_n) \tag{2.20}$$

where $\bar{\alpha}_i \notin \mathcal{Y}$ and $\bar{\alpha}_1 + \bar{\alpha}_2 + \cdots + \bar{\alpha}_n = g$. From the first part, we know that there exist

$$m_1, ..., m_k \in \{0, ..., l'-2\}, \tilde{\beta}_1, ..., \tilde{\beta}_k, \tilde{\delta}_1, ..., \tilde{\delta}_p \in \mathbb{C}/l\mathbb{Z}, \text{ and } W_1, ..., W_p \in \mathscr{C}$$

such that

$$V(0,\tilde{\alpha}_1) \otimes ... \otimes V(0,\tilde{\alpha}_n) \cong_{\mathscr{C}} \left(\bigoplus_u V(m_u,\tilde{\beta}_u)\right) \oplus \left(\bigoplus_t \left(V(l'-1,\tilde{\delta}_t) \otimes W_t\right)\right)$$

where $\bar{\beta}_u, \bar{\delta}_t \notin \{\bar{0}, \bar{\frac{l}{2}}\}$. Applying Lemma 2.5.2.2 to the right side of previous equation there exists a subset $J' \subset \{1, ..., k\}$ and a retract W' of $\oplus_t(V(l'-1,\tilde{\delta}_t) \otimes W_t)$ such that

$$V \simeq_{\mathscr{C}} (\oplus_{u\in J'} V(m_u,\tilde{\beta}_u)) \oplus W'.$$

Now, for all $t \in \{1, ..., p\}$, Corollary 2.5.1.4 implies

$$V(l'-1, \tilde{\delta}_t) \simeq_{\mathscr{C}^N} \{0\}.$$

This shows that

$$\bigoplus_t \Big(V(l'-1, \tilde{\delta}_t) \otimes W_t \Big) \simeq_{\mathscr{C}^N} \{0\}.$$

Using 2.5.2.3, we obtain that:

$$W' \simeq_{\mathscr{C}^N} \{0\}$$

Thus we conclude that in $\mathscr{C}^N$:

$$V \simeq \oplus_{u \in J'} V(m_u, \tilde{\beta}_u). \tag{2.21}$$

Since the modules in the previous decomposition are simples in $\mathscr{C}$, then Lemma 2.5.2.1 implies they are also simple in $\mathscr{C}^N$ and we obtain that V is semi-simple in $\mathscr{C}^N$.

Proof of part 2). Now we will prove the second part of the theorem. Let $V \in \mathscr{C}_g^N$ be a simple object (i.e. $\mathrm{End}_{\mathscr{C}^N}(V) = \mathbb{C}\,\mathrm{Id}_V$), with $g \in G, g \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. By definition V is obtained from a $\mathscr{C}$-retract of tensor products of modules of the form $V(0, \alpha)$ and from Equation (2.21) we have

$$V \simeq_{\mathscr{C}^N} \oplus_{u \in J'} V(m_u, \tilde{\beta}_u)$$

where each $V(m_u, \tilde{\beta}_u)$ is simple in both $\mathscr{C}$ and $\mathscr{C}^N$. If the carnality of J' was strictly greater than one then $dim(\mathrm{End}_{\mathscr{C}^N}(V)) \geq 2$ which is a contradiction. So

$$V \simeq_{\mathscr{C}^N} V(m_u, \tilde{\beta}_u)$$

for some $0 \leq m_u \leq l'-2$ and $\bar{\beta}_i = g$. This shows that any simple object that occur in $\mathscr{C}_g^N$ is of the desired form.

For the other inclusion, let $0 \leq s \leq l'-2, \tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}, \bar{\gamma} = g \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. We will show that $V(s, \tilde{\gamma})$ is in $\mathscr{C}_g^N$.

In the first part of this proof we showed the statements $P(n)$ hold. The last part of these statements imply that for all $0 \leq m \leq l'-2$ and $h \in \mathbb{C}/\mathbb{Z} \setminus \{\bar{0}, \overline{\frac{l}{2}}\}$ there exists $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\beta} = h$ and $V(m, \tilde{\beta})$ is a simple object in $\mathscr{C}_h^N$. We use this as follows.

Choose $\bar{\beta} \in \mathbb{C}/\mathbb{Z}$ such that $\bar{\beta}, \bar{\gamma} - \bar{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. There exists a lift $\tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ of $\bar{\beta}$ so that $V(s, \tilde{\beta})$ is $\mathscr{C}_{\bar{\beta}}^N$ as discussed above. Set $\tilde{\epsilon} = \tilde{\gamma} - \tilde{\beta}$ then $\bar{\epsilon}$ and $\bar{\beta} + \bar{\epsilon} = \bar{\gamma}$ are not in $\{\bar{0}, \overline{\frac{l}{2}}\}$.

In $\mathscr{C}$, by definition $V(s,\tilde{\beta})$ is a retract of a module:

$$V(0,\tilde{\alpha}_1)\otimes V(0,\tilde{\alpha}_2)\otimes ...\otimes V(0,\tilde{\alpha}_n).$$

Taking the tensor product of this module with $V(0,\tilde{\epsilon})$, we get that $V(s,\tilde{\beta})\otimes V(0,\tilde{\epsilon})$ is a $\mathscr{C}$-retract of

$$V(0,\tilde{\alpha}_1)\otimes V(0,\tilde{\alpha}_2)\otimes ...\otimes V(0,\tilde{\alpha}_n)\otimes V(0,\tilde{\epsilon}). \tag{2.22}$$

Since $\bar{\beta},\bar{\epsilon},\bar{\beta}+\bar{\epsilon}\notin\{\bar{0},\frac{\bar{l}}{2}\}$ we have

$$V(s,\tilde{\beta})\otimes V(0,\tilde{\epsilon})\simeq_{\mathscr{C}} V(s,\tilde{\gamma})\oplus V(s+1,\tilde{\gamma})\oplus(1-\tilde{\delta}_{s,0})V(s-1,\tilde{\gamma}+1)\oplus V(s,\tilde{\gamma}+1)$$

and we see that $V(s,\tilde{\gamma})$ is a $\mathscr{C}$-retract of $V(s,\tilde{\beta})\otimes V(0,\tilde{\epsilon})$. Using properties of $\mathscr{C}$-retracts (if A is a retract of B and B is a retract of C, then A is a retract of C) and the previous two decompositions, we have $V(s,\tilde{\gamma})$ is a $\mathscr{C}$-retract of the module in Equation (2.22). This concludes the proof. □

A set of simple objects A is said to be represented by a set of simple objects R_A if any element of A is isomorphic to a unique element of R_A. Lemma 2.5.1.1 and Theorem 2.5.2.4 imply the following corollary.

Corollary 2.5.2.5. *The category $\mathscr{C}^N$ is a generically finitely $\mathbb{C}/\mathbb{Z}$-semi-simple pivotal $\mathbb{C}$-category with small symmetric subset $\mathcal{X}=\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. The class of generic simple objects A of $\mathscr{C}^N$ is represented by*

$$R_{\mathsf{A}}=\{V(n,\tilde{\gamma})\mid 0\leq n\leq l'-2,\tilde{\gamma}\in\mathbb{C}/l\mathbb{Z},\bar{\gamma}\notin\mathcal{X}\}. \tag{2.23}$$

2.5.3 Trace

Here we will show that the right trace t on $\mathcal{I}$ induces a trace in $\mathscr{C}^N$.

Lemma 2.5.3.1. *The full subcategory $\mathcal{I}^N:=\mathscr{C}^N\setminus\{\mathbb{C}\}$ is a right ideal in $\mathscr{C}^N$.*

Proof. We need to show that $\mathcal{I}^N$ satisfies the two conditions to be a right ideal (see Subsection 2.2.4). The first condition is true from the definitions of $\mathscr{C}$ and $\mathscr{C}^N$. For the second condition we need to check that the trivial object $\mathbb{C}$ is not a retract of an object in $\mathscr{C}^N$. On the contrary, suppose there exists an object W in $\mathscr{C}^N$ and morphisms $f\in\mathrm{Hom}_{\mathscr{C}^N}(\mathbb{C},W)$ and $g\in\mathrm{Hom}_{\mathscr{C}^N}(W,\mathbb{C})$ such that $gf=\mathrm{Id}_{\mathbb{C}}$. But by definition $\mathrm{Hom}_{\mathscr{C}^N}(\mathbb{C},W)=\mathrm{Hom}_{\mathscr{C}}(\mathbb{C},W)$ and $\mathrm{Hom}_{\mathscr{C}^N}(W,\mathbb{C})=\mathrm{Hom}_{\mathscr{C}}(W,\mathbb{C})$ so f and g give a $\mathscr{C}$-retract of the trivial module which would imply that $\mathcal{I}=\mathscr{C}$ which is a contradiction to Lemma 2.4.2.2. Thus, $\mathbb{C}\notin\mathscr{C}^N$. □

Lemma 2.5.3.2. *For $V \in \mathcal{I}^N$ the assignment $\mathsf{t^N}_V : \mathrm{End}_{\mathscr{C}^N}(V) \to \mathbb{C}$ given by $[f] \mapsto \mathsf{t}_V(f)$ is a well defined linear function. Moreover, the family $\{\mathsf{t^N}_V\}_{V\in\mathcal{I}^N}$ is a right trace on $\mathcal{I}^N$.*

Proof. We need to show $\mathsf{t^N}_V$ does not depend on the representative of $[f]$ in

$$\mathrm{End}_{\mathscr{C}^N}(V) = \mathrm{Hom}_{\mathscr{C}^N}(V, V) = \mathrm{Hom}_{\mathscr{C}}(V, V)/Negl(V, V).$$

Suppose $[f] = [g]$ then $f = g + h$ for some $h \in Negl(V, V)$. Then $\mathsf{t}_V(f) = \mathsf{t}_V(g + h) = \mathsf{t}_V(g)$, implying $\mathsf{t^N}_V([f]) = \mathsf{t^N}_V([g])$.

In order to prove that $\mathsf{t^N}_V$ is a right trace on $\mathcal{I}^N$, we have to prove that this satisfies the conditions 1) and 2) from the definition.

1) Let $U, V \in \mathcal{I}^N$ and $[f] \in \mathrm{Hom}_{\mathscr{C}^N}(V, U), [g] \in \mathrm{Hom}_{\mathscr{C}^N}(U, V)$. Let $f \in \mathrm{Hom}_{\mathscr{C}}(V, U), g \in \mathrm{Hom}_{\mathscr{C}}(U, V)$ such that the class of f and g in $\mathscr{C}^N$ are $[f]$ and $[g]$ respectively. Then

$$\mathsf{t^N}_V([g][f]) = \mathsf{t^N}_V([gf]) = \mathsf{t}_V(gf) = \mathsf{t}_U(fg) = \mathsf{t^N}_U([fg]) = \mathsf{t^N}_U([f][g]).$$

2) Consider $U \in \mathcal{I}^N$ and $W \in \mathscr{C}$ and $f \in \mathrm{End}^N_{\mathscr{C}}(U \otimes W)$. Let $f \in \mathrm{End}_{\mathscr{C}}(U \otimes W)$ such that the class of f in $\mathscr{C}$ is $[f]$. Then we obtain:

$$\mathsf{t^N}_{U\otimes W}([f]) = \mathsf{t}_{U\otimes W}(f) = \mathsf{t}_U\Big((\mathrm{Id}_U \otimes \overleftarrow{\mathrm{ev}}_W)(f \otimes \mathrm{Id}_{W^*})(\mathrm{Id}_U \otimes \overrightarrow{\mathrm{coev}}_W)\Big) =$$

$$= \mathsf{t^N}_U\Big(([\mathrm{Id}_U] \otimes [\overleftarrow{\mathrm{ev}}_W])([f] \otimes [\mathrm{Id}_{W^*}])([\mathrm{Id}_U] \otimes [\overrightarrow{\mathrm{coev}}_W)]\Big)$$

The previous two equalities conclude the statement. □

2.5.4 T-ambi pair

Let $\mathsf{d} : R_{\mathsf{A}} \to \mathbb{C}$ be the function given in Equation (2.15), in other words:

$$\mathsf{d}(V(n, \tilde{\alpha})) = \frac{\{n + 1\}}{\{1\}\{\tilde{\alpha}\}\{\tilde{\alpha} + n + 1\}} \tag{2.24}$$

for $V(n, \tilde{\alpha}) \in R_{\mathsf{A}}$. Extend this function to A by requiring $\mathsf{d}(V) = \mathsf{d}(V(n, \tilde{\alpha}))$ if V is isomorphic to $V(n, \tilde{\alpha})$.

Lemma 2.5.4.1. *The pair (A, d) is a t-ambi pair in $\mathscr{C}^N$.*

Proof. Consider d^{N} the modified dimension on $\mathcal{I}^N$ coming from the right trace t^{N} on $\mathscr{C}^N$.

Let $\mathsf{B} := \{V \in \mathcal{I}^N \bigcap (\mathcal{I}^N)^* \mid V \text{ simple},\ \mathsf{d}^{\mathrm{N}}(V) = \mathsf{d}^{\mathrm{N}}(V^*)\}$. From Theorem 2.2.4.4, it follows that $(\mathsf{B}, \mathsf{d}^N)$ is a t-ambi pair. We notice that $(\mathcal{I}^N)^* = \mathcal{I}^N$.

We will prove that $\mathsf{A} \subseteq \mathsf{B}$ and that d^{N} is determined by Equation (2.24). Let $V \in \mathsf{A}$. By definition there exists $0 \leq n \leq l' - 2$ and $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ with $\bar{\gamma} \notin \mathcal{X}$ such that $V \simeq V(n, \tilde{\gamma})$. We have

$$\mathsf{d}^{\mathrm{N}}(V(n,\tilde{\gamma})) = \mathsf{t}^{\mathrm{N}}([\mathrm{Id}_V(n,\tilde{\gamma})]) = \mathsf{t}(\mathrm{Id}_V(n,\tilde{\gamma})) = \mathsf{d}(V(n,\tilde{\gamma}))$$

and so $\mathsf{d}^{\mathrm{N}}(V(n,\tilde{\gamma}))$ is given by the formula in Equation (2.24).

Since $V(n,\tilde{\alpha})^* = V(n, -\tilde{\alpha} - \tilde{n} - \tilde{1})$, Equation (2.24) implies

$$\mathsf{d}(V(n,\tilde{\alpha})) = \mathsf{d}((V(n,\tilde{\alpha}))^*).$$

We conclude that

$$\mathsf{d}^{\mathrm{N}}(V(n,\tilde{\alpha})) = \mathsf{d}^{\mathrm{N}}((V(n,\tilde{\alpha}))^*)$$

for any $V \in \mathsf{A}$. This shows that $\mathsf{A} \subseteq \mathsf{B}$. Thus, since $(\mathsf{B}, \mathsf{d}^{\mathrm{N}})$ is a t-ambi pair, it is easy to check that (A, d) is a t-ambi pair. □

2.5.5 The b map

Here we show $\mathscr{C}^N$ has a map b as in the definition of a relative G-spherical category. To do this we need the following technical lemmas.

Lemma 2.5.5.1. *Let $L, R \in \mathscr{C}^N$ with $L \in \mathscr{C}^N_g$, $R \in \mathscr{C}^N_h$ with $g, h \notin \{\bar{0}, \frac{\bar{l}}{2}\}$. Suppose $L =_{\mathscr{C}^N} L_1 \oplus L_2$ and $R =_{\mathscr{C}^N} R_1 \oplus R_2$ such that*

$$L \simeq_{\mathscr{C}^N} R \quad \text{and} \quad L_1 \simeq_{\mathscr{C}^N} R_1.$$

Then

$$L_2 \simeq_{\mathscr{C}^N} R_2.$$

Proof. Using Theorem 2.5.2.4, we have that both L and R are semi-simple in $\mathscr{C}^N$. More precisely, there exists $N \in \mathbb{N}$, and $S_1, ..., S_N \in R_{\mathsf{A}}$ all different such that:

$$L = \eta_1 S_1 \oplus ... \oplus \eta_N S_N$$

$$R = \eta'_1 S_1 \oplus ... \oplus \eta'_N S_N \oplus J$$

where $\eta_i, \eta_i' \in \mathbb{N}$, are the multiplicities of the simple object S_i and J is a direct sum of elements of R_A which are all different than $S_i, i \in \{1, ..., N\}$. As an observation, from the computation of d we have:

$$\mathsf{d}(V) \neq 0 \qquad \text{for all} \qquad V \in R_A,$$

in particular $\mathsf{d}(S_i) \neq 0$, for all $i \in \{1, ..., N\}$.

We have that:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(L, R) &= \mathrm{Hom}_{\mathscr{C}}(\eta_1 S_1 \oplus ... \oplus \eta_N S_N, \eta_1' S_1 \oplus ... \oplus \eta_N' S_N \oplus J) \\ &= \bigoplus_{i,j} (\mathrm{Hom}_{\mathscr{C}}(\eta_i S_i, \eta_j' S_j)) \oplus \bigoplus_i (\mathrm{Hom}_{\mathscr{C}}(\eta_i S_i, J)).\end{aligned}$$

We notice that $\mathrm{Hom}(S_i, S_j) = 0$ for $i \neq j$ and $\mathrm{Hom}(S_i, J) = 0$ since J has no S_i-isotipic components so:

$$\mathrm{Hom}_{\mathscr{C}}(L, R) = \bigoplus_i (\mathrm{Hom}_{\mathscr{C}}(\eta_i S_i, \eta_i' S_i)).$$

Now we will study the negligible morphisms from this space. By definition $\mathrm{Negl}(L, R) \subseteq \mathrm{Hom}_{\mathscr{C}}(L, R)$ as a vector subspace. From the last two relations we obtain that:

$$\mathrm{Negl}(L, R) = \bigoplus_i (\mathrm{Negl}(\eta_i S_i, \eta_i' S_i)).$$

We will prove that actually we have no negligible morphisms between isotipic components of S_i.

Suppose that there exists $f \in Negl(\eta_i S_i, \eta_i' S_i)$ which is non-zero. For $k \in \{1, ..., N\}$, denote by $\iota_k : S_i \to \eta_i S_i$ and $\pi_k : \eta_i' S_i \to S_i$ the inclusion and projection of the k^{th} component.

Since f is non-zero, then there exists $k, l \in \{1, ..., N\}$ such that $\pi_l \circ f \circ \iota_k \neq 0$. Also, since S_i is simple in $\mathscr{C}$:

$$\pi_l \circ f \circ \iota_k = \pi_l \circ f \circ \iota_k(1)\, \mathrm{Id}_{S_i}.$$

At the level of the modified trace we have:

$$\mathsf{t}_{S_i}(\pi_l \circ f \circ \iota_k) = (\pi_l \circ f \circ \iota_k(1))\, \mathsf{t}_{S_i}(\mathrm{Id}_{S_i}) = (\pi_l \circ f \circ \iota_k(1))\, \mathsf{d}(S_i) \neq 0.$$

From the properties of t, we have:

$$\mathsf{t}_{S_i}(\pi_l \circ f \circ \iota_k) = \mathsf{t}_{\eta_i' S_i}(f \circ \iota_k \circ \pi_l) = 0$$

(since f is negligible).

The last two equalities lead to a contradiction. We conclude that $Negl(L,R) = \{0\}$ and so:

$$\mathrm{Hom}_{\mathscr{C}}(L,R) = \mathrm{Hom}_{\mathscr{C}^N}(L,R).$$

Now let $[\phi] \in \mathrm{Hom}_{\mathscr{C}^N}(L,R)$ be an isomorphism. Consider $\phi \in \mathrm{Hom}_{\mathscr{C}}(L,R)$ that gives $[\phi]$ in $\mathscr{C}^N$. From the previous considerations, $\phi : L \to R$ is an isomorphism in $\mathscr{C}$.

Using this, we obtain that $J = \{0\}$ (it is not possible to have more isotipic components in R than in L). Also, since we are in a category of representations which are semi-simple and morphisms between representations, we obtain that

$$\eta_i = \eta_i', \forall i \in \{1, ..., N\}.$$

So now, both R and L are semi-simple modules in $\mathscr{C}$ with the same isotipic decomposition.

Now both L_1 and R_1 are direct summands in L and R. It means that each of them has a semi-simple decomposition with modules from the set S_i. But L_1 and R_1 are isomorphic in $\mathscr{C}^N$. Using the same argument as in the first part with L and R, we obtain that L_1 and R_1 are isomorphic in $\mathscr{C}$. It means that they have the same isotipic decompositions with the same multiplicities. Let us compose ϕ to the right with an automorphism of L that makes a permutation on the isotipic components such that the ones corresponding to L_1 are sent onto the ones corresponding to $\phi^{-1}(R_1)$ respectively. This means that we obtain an isomorphism

$$\tilde{\phi} : L \to R$$

such that

$$\tilde{\phi}(L_1) = R_1.$$

We conclude that

$$\tilde{\phi}|_{L_2} : L_2 \to R_2$$

is an isomorphism in $\mathscr{C}$ and also in $\mathscr{C}^N$. □

Lemma 2.5.5.2. *For all $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ and $n \in \mathbb{N}$ such that $\bar{\alpha}, \bar{\beta}, \bar{\alpha}+\bar{\beta} \notin \{\bar{0}, \overline{\tfrac{l}{2}}\}$ and $n \leq l'-2$ then given $m \leq n$ we have*

$$V(m,\tilde{\alpha})\otimes V(n,\tilde{\beta}) \simeq_{\mathscr{C}^N} V(0,\tilde{\alpha})\otimes\Big(V(n+m,\tilde{\beta})\oplus V(n+m-2,\tilde{\beta}+1)\oplus\cdots\oplus V(n-m,\tilde{\beta}+m)\Big)$$

where we set $V(k,\tilde{\beta}) = 0$ if $k \geq l'-1$.

Proof. We will show the statement by induction on m. The case $m = 0$ is true from Lemma 2.3.4.5. Next, we will check the case $m = 1$.

Let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}/l\mathbb{Z}$ and $n \in \mathbb{N}$ such that $\bar{\alpha}, \bar{\beta}, \bar{\alpha} + \bar{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$ and $1 \leq n \leq l' - 2$. Choose $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\gamma}, \bar{\alpha} - \bar{\gamma}, \bar{\alpha} - \bar{\gamma} + \bar{\beta} \notin \{\bar{0}, \overline{\frac{l}{2}}\}$. From Lemma 2.3.4.5 we have

$$\begin{aligned} V(0, \tilde{\alpha} - \tilde{\gamma}) \otimes V(n, \tilde{\beta}) \simeq_{\mathscr{C}^N} & V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta}) \oplus V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1) \\ & \oplus (1 - \delta_{l'-2,n}) V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta}) \oplus V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1) \end{aligned}$$

in $\mathscr{C}^N$. Take the tensor product of both sides of this equation with $V(0, \tilde{\gamma})$. Then decomposing the left side by grouping the first two simple modules together we have

$$\begin{aligned} \Big(V(0, \tilde{\gamma}) \otimes V(0, \tilde{\alpha} - \tilde{\gamma})\Big) \otimes V(n, \tilde{\beta}) \simeq_{\mathscr{C}^N} & \Big(V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})\Big) \\ & \oplus \Big(V(0, \tilde{\alpha} + 1) \otimes V(n, \tilde{\beta})\Big) \oplus \Big(V(1, \tilde{\alpha}) \otimes V(n, \tilde{\beta})\Big) \end{aligned}$$

On the other hand the left side is

$$\begin{aligned} & \Big(V(0, \tilde{\gamma}) \otimes V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})\Big) \oplus \Big(V(0, \tilde{\gamma}) \otimes V(n, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)\Big) \\ & \quad \oplus (1 - \delta_{l'-2,n}) \Big(V(0, \tilde{\gamma}) \otimes V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})\Big) \\ & \quad \quad \oplus \Big(V(0, \tilde{\gamma}) \otimes V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)\Big) \end{aligned}$$

Now, using Corollary 2.3.4.6 we see the first two tensor products in the last two expressions are the same direct sum of simple modules so they are isomorphic. Thus, Lemma 2.5.5.1 implies

$$\begin{aligned} V(1, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathscr{C}^N} & (1 - \delta_{l'-2,n}) \Big(V(0, \tilde{\gamma}) \otimes V(n+1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta})\Big) \\ & \oplus \Big(V(0, \tilde{\gamma}) \otimes V(n-1, \tilde{\alpha} - \tilde{\gamma} + \tilde{\beta} + 1)\Big). \end{aligned}$$

Using Corollary 2.3.4.6 we see the right hand side of this equation is isomorphic to

$$(1 - \delta_{l'-2,n}) \Big(V(0, \tilde{\alpha}) \otimes V(n+1, \tilde{\beta})\Big) \oplus \Big(V(0, \tilde{\alpha}) \otimes V(n-1, \tilde{\beta} + 1)\Big).$$

Thus, we have proved the lemma for the case $m = 1$.

Now assume the statement is true for $k \leq m$ and we will show the statement holds for $m+1$. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and n be as above. Let us denote:

$$E^{\tilde{\beta}}_{m,n} := \Big(V(n+m,\tilde{\beta}) \oplus V(n+m-2,\tilde{\beta}+1) \oplus \cdots \oplus V(n-m,\tilde{\beta}+m)\Big).$$

From the induction hypothesis we have

$$V(0,\tilde{\gamma}) \otimes V(m,\tilde{\alpha}-\tilde{\gamma}) \otimes V(n,\tilde{\beta}) \simeq_{\mathscr{C}^N} V(0,\tilde{\gamma}) \otimes V(0,\tilde{\alpha}-\tilde{\gamma}) \otimes E^{\tilde{\beta}}_{m,n}. \quad (2.25)$$

Using Lemma 2.3.4.5 to decompose the first tensor product we have the left hand side of Equation (2.25) is isomorphic to

$$\Big(V(m,\tilde{\alpha}) \otimes V(n,\tilde{\beta})\Big) \oplus \Big(V(m,\tilde{\alpha}+1) \otimes V(n,\tilde{\beta})\Big) \oplus$$

$$\oplus \Big(V(m-1,\tilde{\alpha}+1) \otimes V(n,\tilde{\beta})\Big) \oplus \Big(V(m+1,\tilde{\alpha}) \otimes V(n,\tilde{\beta})\Big)$$

Similarly, the right hand side of Equation (2.25) is isomorphic to

$$\Big(V(0,\tilde{\alpha}) \otimes E^{\tilde{\beta}}_{m,n}\Big) \oplus \Big(V(0,\tilde{\alpha}+1) \otimes E^{\tilde{\beta}}_{m,n}\Big) \oplus \Big(V(1,\tilde{\alpha}) \otimes E^{\tilde{\beta}}_{m,n}\Big)$$

From the induction hypothesis, the first two terms of the last two expressions are isomorphic, thus from Lemma 2.5.5.1 we obtain

$$\Big(V(m-1,\tilde{\alpha}+1) \otimes V(n,\tilde{\beta})\Big) \oplus \Big(V(m+1,\tilde{\alpha}) \otimes V(n,\tilde{\beta})\Big) \simeq_{\mathscr{C}^N} V(1,\tilde{\alpha}) \otimes E^{\tilde{\beta}}_{m,n}. \quad (2.26)$$

Next, we decompose the right side of this equation. By induction we know that for any $1 \leq n' \leq l'-2$ and $k \leq m$ we have

$$V(1,\tilde{\alpha}) \otimes V(n',\tilde{\beta}+k) \simeq_{\mathscr{C}^N} V(0,\tilde{\alpha}) \otimes \Big(V(n'+1,\tilde{\beta}+k) \oplus V(n'-1,\tilde{\beta}+k+1)\Big).$$

Applying this to each term of the sum $E^{\tilde{\beta}}_{m,n}$ we obtain:

$$V(1,\tilde{\alpha}) \otimes E^{\tilde{\beta}}_{m,n} \simeq_{\mathscr{C}^N} V(0,\tilde{\alpha}) \otimes \left(E^{\tilde{\beta}}_{m,n+1} \oplus E^{\tilde{\beta}+1}_{m,n-1}\right)$$

From the definition $E^{\tilde{\beta}}_{m,n}$ we have

$$\begin{aligned} V(0,\tilde{\alpha})\otimes &\left(E^{\tilde{\beta}}_{m,n+1}\oplus E^{\tilde{\beta}+1}_{m,n-1}\right) = \\ &= V(0,\tilde{\alpha})\otimes\left(E^{\tilde{\beta}}_{m,n+1}\oplus\left(E^{\tilde{\beta}+1}_{m-1,n}\oplus V(n-1-m,\tilde{\beta}+m+1)\right)\right) \\ &= V(0,\tilde{\alpha})\otimes\left(\left(E^{\tilde{\beta}}_{m,n+1}\oplus V(n-1-m,\tilde{\beta}+m+1)\right)\oplus E^{\tilde{\beta}+1}_{m-1,n}\right) \\ &= V(0,\tilde{\alpha})\otimes\left(E^{\tilde{\beta}}_{m+1,n}\oplus E^{\tilde{\beta}+1}_{m-1,n}\right) \\ &\simeq_{\mathscr{C}^N}\left(V(0,\tilde{\alpha})\otimes E^{\tilde{\beta}}_{m+1,n}\right)\oplus\left(V(m-1,\tilde{\alpha}+1)\otimes V(n,\tilde{\beta})\right) \end{aligned}$$

where the isomorphism comes from the induction hypothesis. Combining the last two equation and using Lemma 2.5.5.1 we see that Equation (2.26) implies

$$V(m+1,\tilde{\alpha})\otimes V(n,\tilde{\beta})\simeq_{\mathscr{C}^N} V(0,\tilde{\alpha})\otimes E^{\tilde{\beta}}_{m+1,n}$$

which proves the statement for $m+1$ and concludes the induction step. □

Now we use this lemma to show $\mathscr{C}^N$ has a b map. In [34] it is shown how to construct a b map from a character. Our b map will be defined on the representative class of simple objects R_{A} and extended to A by setting $\mathsf{b}(W)=\mathsf{b}(V)$ if $W\simeq V$ for $W\in\mathsf{A}$ and $V\in R_{\mathsf{A}}$.

Here a character is a map $\chi: R_{\mathsf{A}}\to\mathbb{C}$ satisfying

1. $\chi(V^*)=\chi(V)$ for all $V\in R_{\mathsf{A}}$,
2. if $V(m,\tilde{\alpha}), V(n,\tilde{\beta})\in R_{\mathsf{A}}$ such that $\bar{\alpha}+\bar{\beta}\notin\mathcal{X}$ then

   $$\chi(m,\tilde{\alpha})\chi(n,\tilde{\beta})=\sum_{k,\tilde{\gamma}}\dim\left(\mathrm{Hom}_{\mathscr{C}^N}(V(k,\tilde{\gamma}),V(m,\tilde{\alpha})\otimes V(n,\tilde{\beta}))\right)\chi(k,\tilde{\gamma})$$

   here for simplicity we denoted $\chi(V(m,\tilde{\alpha}))=\chi(m,\tilde{\alpha})$,
3. for any $g\in G\backslash\mathcal{X}$, the element $\mathcal{D}_g=\sum_{V\in\mathscr{C}^N_g\cap R_{\mathsf{A}}}\chi(V)^2$ of $\mathbb{C}$ is non-zero.

If χ is a character then Lemma 23 of [34] implies the map $G\backslash\mathcal{X}\to\mathbb{C}$, $g\mapsto\mathcal{D}_g$ is a constant function with value $\mathcal{D}$. Moreover, the map $\mathsf{b}=\frac{1}{\mathcal{D}}\chi$ satisfies the properties listed in Definition 2.2.3.1. We will now show a character exists.

Let $V(m,\tilde{\alpha})$ be in R_{A}. Consider the formal character $\chi(m,\tilde{\alpha})=\sum_{k,\tilde{\gamma}}c_{k,\tilde{\gamma}}e^k e^{\tilde{\gamma}}$ of $V(m,\tilde{\alpha})$ in $\mathscr{C}$. Here e^k and $e^{\tilde{\gamma}}$ are both formal variables for each $k\in\mathbb{Z}$

and $\tilde{\gamma} \in \mathbb{C}/l\mathbb{Z}$ and $c_{k,\tilde{\gamma}}$ is the dimension of the $(k, \tilde{\gamma})$ weight space determined by the action of (K_1, K_2).

The variables e^k and $e^{\tilde{\gamma}}$ of a character $\chi(m, \tilde{\alpha})$ can be specialized to q^k and 1, respectively to obtain a complex number which we denote by $\chi_q(m, \tilde{\alpha}) \in \mathbb{C}$. We will show χ_q is a character in $\mathscr{C}^N$.

Using the basis in Theorem 2.3.4.3 we see

$$\chi_q(m, \tilde{\alpha}) = (2 + q + q^{-1})(q^m + q^{m-2} + \cdots + q^{-m}) = (2 + q + q^{-1})[m + 1].$$

In particular, $\chi_q(l' - 1, \tilde{\alpha}) = 0$. Also, $V(m, \tilde{\alpha})^* = V(m, -\tilde{\alpha} - \tilde{m} - \tilde{1})$ which implies $\chi_q(V^*) = \chi_q(V)$ for $V \in R_{\mathsf{A}}$.

Next, we show property (2) holds for χ_q. We first do this for the case $m = 0$ and $0 \leq n \leq l' - 2$. Let $V(0, \tilde{\alpha}), V(n, \tilde{\beta}) \in R_{\mathsf{A}}$ such that $\bar{\alpha} + \bar{\beta} \notin \mathcal{X}$. From Lemma 2.3.4.5 we have

$$\begin{aligned} V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathscr{C}} & V(n, \tilde{\alpha} + \tilde{\beta}) \oplus V(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ & \oplus (1 - \delta_{0,n}) V(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(n, \tilde{\alpha} + \tilde{\beta} + 1) \end{aligned} \tag{2.27}$$

in $\mathscr{C}$. This implies

$$\chi(0, \tilde{\alpha})\chi(n, \tilde{\beta}) = \chi(n, \tilde{\alpha}+\tilde{\beta})+\chi(n+1, \tilde{\alpha}+\tilde{\beta})+(1-\delta_{0,n})\chi(n-1, \tilde{\alpha}+\tilde{\beta}+1)+\chi(n, \tilde{\alpha}+\tilde{\beta}+1).$$

By specializing the variables of this equation we have:

$$\begin{aligned} \chi_q(0, \tilde{\alpha})\chi_q(n, \tilde{\beta}) = & \chi_q(n, \tilde{\alpha} + \tilde{\beta}) + \chi_q(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ & + (1 - \delta_{0,n})\chi_q(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) + \chi_q(n, \tilde{\alpha} + \tilde{\beta} + 1). \end{aligned} \tag{2.28}$$

Translating Equation (2.27) to $\mathscr{C}^N$ we have:

$$\begin{aligned} V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta}) \simeq_{\mathscr{C}^N} & V(n, \tilde{\alpha} + \tilde{\beta}) \oplus (1 - \delta_{l'-2,n}) V(n + 1, \tilde{\alpha} + \tilde{\beta}) \\ & \oplus (1 - \delta_{0,n}) V(n - 1, \tilde{\alpha} + \tilde{\beta} + 1) \oplus V(n, \tilde{\alpha} + \tilde{\beta} + 1). \end{aligned}$$

Since $\chi_q(l' - 1, \tilde{\alpha} + \tilde{\beta}) = 0$ then the last equation implies we can rewrite Equation (2.28) as

$$\chi_q(0, \tilde{\alpha})\chi_q(n, \tilde{\beta}) = \sum_{k,\tilde{\gamma}} \dim \left(\mathrm{Hom}_{\mathscr{C}^N}(V(k, \tilde{\gamma}), V(0, \tilde{\alpha}) \otimes V(n, \tilde{\beta})) \right) \chi_q(k, \tilde{\gamma})$$

here all but possibly four homomorphism spaces are zero. This implies that if $W \simeq_{\mathscr{C}^N} \bigoplus_{i=1}^{s} \left(V(0,\tilde{\alpha}) \otimes V(n_i,\tilde{\beta}_i) \right)$ where $\bar{\alpha}, \bar{\beta}_i, \bar{\alpha}+\bar{\beta}_i \notin \mathcal{X}$ and $0 \leq n_i \leq l'-2$ for all $i \in \{1,...,s\}$ then

$$\sum_{i=1}^{s} \chi_q(0,\tilde{\alpha})\chi_q(n_i,\tilde{\beta}_i) = \sum_{k,\tilde{\gamma}} \dim\left(\mathrm{Hom}_{\mathscr{C}^N}(V(k,\tilde{\gamma}),W)\right)\chi_q(k,\tilde{\gamma}). \qquad (2.29)$$

Next we consider the general case. Let $V(m,\tilde{\alpha}), V(n,\tilde{\beta}) \in R_{\mathsf{A}}$ such that $\bar{\alpha}+\bar{\beta} \notin \mathcal{X}$. A direct computation shows

$$\chi_q(m,\tilde{\alpha})\chi_q(n,\tilde{\beta}) = (2+q+q^{-1})^2[m+1][n+1].$$

where it can be shown that

$$[m+1][n+1] = [n+1+m] + [n+1+m-2] + ... + [n+1-m].$$

This implies,

$$\begin{aligned} &\chi_q(m,\tilde{\alpha})\chi_q(n,\tilde{\beta}) \\ &= \chi_q(0,\tilde{\alpha})\left(\chi_q(n+m,\tilde{\beta}) + \chi_q(n+m-2,\tilde{\beta}+1) + ... + \chi_q(n-m,\tilde{\beta}+m)\right). \end{aligned}$$

But from Lemma 2.5.5.2 and Equation (2.29) we have the right side of the last equation is equal to

$$\sum_{k,\tilde{\gamma}} \dim\left(\mathrm{Hom}_{\mathscr{C}^N}(V(k,\tilde{\gamma}), V(m,\tilde{\alpha}) \otimes V(n,\tilde{\beta}))\right)\chi_q(k,\tilde{\gamma})$$

and we have shown that property (2) holds.

Finally, we show χ_q satisfies the last property to be a character. Fix $\tilde{\alpha} \in \mathbb{C}/l\mathbb{Z}$ such that $\bar{\alpha} = g \notin \mathcal{X}$ then for any $k \in \{0,...,l'-1\}$ we have

$$\sum_{m=0}^{l'-1} \chi_q(m,\tilde{\alpha}+k)^2 = c^2 \sum_{m=0}^{l'-1} [m+1]^2 = \frac{c^2}{(q-q^{-1})^2} \sum_{m=0}^{l'-1} (q^{m+1} - q^{-m-1})^2$$

where $c = 2 + q + q^{-1}$. Let us compute the sum in this expression:

$$\begin{aligned}\sum_{m=0}^{l'-1}(q^{m+1} - q^{-m-1})^2 &= \sum_{m=0}^{l'-1}\left(q^{2m+2} + q^{-2m-2} - 2\right)\\ &= -2l' + q^2\sum_{m=0}^{l'-1} q^{2m} + q^{-2}\sum_{m=0}^{l'-1} q^{-2m}\\ &= -2l' + q^2\frac{q^{2l'}-1}{q^2-1} + q^{-2}\frac{q^{-2l'}-1}{q^{-2}-1} = -2l'.\end{aligned}$$

Thus, we have

$$\mathcal{D}_g = \sum_{V\in\mathscr{C}_g^N\cap R_{\mathsf{A}}} \chi_q(V)^2 = \sum_{m,k=0}^{l'-1}\chi_q(m,\tilde{\alpha}+k)^2 = \sum_{k=0}^{l'-1}\frac{-2l'c^2}{(q-q^{-1})^2} = \frac{-2(l')^2c^2}{(q-q^{-1})^2}$$

which is non-zero.

In summary, χ_q is a character and as explained above leads to a map $\mathsf{b} = \frac{1}{\mathcal{D}}\chi$ satisfies the properties listed in Definition 2.2.3.1.

2.5.6 Main theorem

Here we summarize the results of this chapter in the following theorem.

Theorem 2.5.6.1. *Let $G = \mathbb{C}/\mathbb{Z}$ and $\mathcal{X} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Let A be the set of generic simple objects of $\mathscr{C}^N$ given in Equation* (2.23)*. Let $\mathsf{d} : \mathsf{A} \to \mathbb{C}^\times$ be the function defined in Equation* (2.24)*. Let $\mathsf{b} : \mathsf{A} \to \Bbbk$ be the function constructed in Subsection 2.5.5. With this data $\mathscr{C}^N$ is a relative G-spherical category with basic data and leads to the modified TV-invariant described in Theorem 2.1.0.1.*

Proof. The proof follows directly from Corollary 2.5.2.5 and Lemmas 2.5.4.1 and 2.2.3.2. □

Table 2.1: Action on $V_1(\tilde{\alpha})$, where $c = q^{-\tilde{\alpha}}(q^{-1}[\tilde{\alpha}] - [\tilde{\alpha}+1])$.

$V_1(\tilde{\alpha})$	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
E_1	0	0	v_1	0	0	v_4	0	0
E_2	0	$c \cdot v_0$	0	0	$[\tilde{\alpha}][\tilde{\alpha}+1]v_3$	0	v_5	$-[\tilde{\alpha}][\tilde{\alpha}+1]v_2$
F_1	0	v_2	0	0	v_5	0	0	0
F_2	v_1	0	v_3	0	0	$-c \cdot v_6$	0	v_4

Table 2.2: Action on $V_2(\tilde{\alpha})$, where $c = q^{-\tilde{\alpha}}(q^{-1}[\tilde{\alpha}] - [\tilde{\alpha}+1])$.

$V_2(\tilde{\alpha})$	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7
E_1	0	u_0	u_5	u_6	u_7	0	$(q+q^{-1})v_5$	0
E_2	0	0	0	0	$-c \cdot u_3$	$c \cdot u_0$	$c \cdot u_1$	$c \cdot u_2$
F_1	u_1	0	u_3	0	0	u_6	0	u_4
F_2	u_5	u_2	0	u_4	0	0	u_7	0

Chapter 3

A combinatorial description of the centralizer algebras connected to the Links-Gould Invariant

In this chapter, we study the tensor powers of a 4-dimensional representation of the quantum super-algebra $U_q(sl(2|1)$, focusing on the rings of its algebra endomorphisms so called centralizer algebras, denoted by LG_n. Their dimensions were conjectured by Marin and Wagner [67]. We will prove this conjecture, describing the intertwiners spaces from a semi-simple decomposition as sets consisting in certain paths in a planar lattice with integer coordinates.

Structure of the chapter

This part is split into four main sections. In Part 3.1, we present the super quantum group $U_q(sl(2|1))$ and some properties concerning its representation theory. Further on, in Section 3.2 we discuss the definition of the Links-Gould invariant. Section 3.3 is devoted to the definition of the centralizer algebras $LG_n(\alpha)$ and some properties and conjectures about them. In the last part, Section 3.4, we prove the Marin-Wagner Conjecture concerning the dimensions of these centralizer algebras.

3.1 The quantum group $U_q(sl(2|1))$

In this section we will introduce the super-quantum group $U_q(sl(2|1))$ and discuss about its representation theory. Let us fix a ground field $\Bbbk$. A super vector space is a vector space over $\Bbbk$ with a $\mathbb{Z}_2$ grading: $V = V_0 \oplus V_1$. A homogenous element $x \in V$ is called even if $x \in V_0$ and odd if $x \in V_1$. Through this section, all the objects that we will work with will respect the $\mathbb{Z}_2$-gradings.

Notation 3.1.0.1. *Consider $\hbar$ to be an indeterminate and the field $\Bbbk := \mathbb{C}((\hbar))$. Denote by:*

$$q := e^{\frac{\hbar}{2}} \qquad \{x\} := q^x - q^{-x}$$

$$exp_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!} \qquad (k)_q = \frac{1-q^k}{1-q} \qquad (n)_q! = (1)_q(2)_q...(n)_q$$

For two homogeneous elements x and y with gradings $\bar{x}, \bar{y} \in \mathbb{Z}_2$, the super-commutator has the formula:

$$[x, y] := xy - (-1)^{\bar{x}\bar{y}} yx$$

Definition 3.1.0.2. *Let $A = (a_{ij})$ be the square matrix given by*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

Consider $U_q(sl(2|1))$ to be the superalgebra over $\mathbb{C}((\hbar))$ generated by $\{E_i, F_i, h_i\}_{i\in\{1,2\}}$ where the generators E_2 and F_2 are odd and all the others are even, with the following relations:

$$[h_i, h_j] = 0, \qquad [h_i, E_j] = a_{i,j} E_j, \qquad [h_i, F_j] = -a_{i,j} F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \qquad E_2^2 = F_2^2 = 0$$

$$E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1^2 = 0, \quad F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1^2 = 0 \tag{3.1}$$

The algebra $U_q(sl(2|1))$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned} \Delta(E_i) =& E_i \otimes 1 + q^{-h_i} \otimes E_i, & \epsilon(E_i) =& 0 & S(E_i) =& -q^{h_i} E_i \\ \Delta(F_i) =& F_i \otimes q^{h_i} + 1 \otimes F_i, & \epsilon(F_i) =& 0 & S(F_i) =& -F_i q^{-h_i} \\ \Delta(h_i) =& h_i \otimes 1 + 1 \otimes h_i, & \epsilon(h_i) =& 0 & S(h_i) =& -h_i. \end{aligned}$$

Notation 3.1.0.3. *Consider the following elements:*

$$E' := E_1 E_2 - q^{-1} E_2 E_1 \qquad F' = F_2 F_1 - q F_1 F_2$$

Proposition 3.1.0.4. *In [50],[84] it has been shown that the quantum group $U_q(sl(2|1))$ admits an R-matrix, defined in the following way: $R = \check{R}K$ where:*

$$\check{R} = exp_q(\{1\} E_1 \otimes F_1) exp_q(-\{1\} E' \otimes F') exp_q(-\{1\} E_2 \otimes F_2)$$
$$K = q^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}$$

In this sequel we will recall some facts about the representation theory of the quantum group $U_q(sl(2|1))$. We would like to emphasyse the fact that the representation theory of super-quantum groups has continuous families of representations already when q is generic. This is in contrast with the classical case of quantum groups, where it is possible to gain a continuous family of representations, just if q is specialised to a root of unity. We will follow [28].

Definition 3.1.0.5. *1) An element $v \in V$ is called a weight vector of weight $\lambda = (\alpha_1, \alpha_2)$ if:*

$$h_i v = \alpha_i v, \forall\ i \in \{1, 2\}.$$

2) A weight vector is called highest weight vector if:

$$E_i v = 0, \forall i \in \{1, 2\}.$$

3)A module V is called a highest weight module of weight $\lambda = (\alpha_1, \alpha_2)$ if it is generated by a highest weight vector $v_0 \in V$ of weight λ.

Proposition 3.1.0.6. *There exists a continuous family of simple representations of $U_q(sl(2|1))$,*
indexed by $\Lambda = \mathbb{N} \times \mathbb{C}$:

$$\lambda = (\alpha_1, \alpha_2) \in \Lambda \longleftrightarrow V(\alpha_1, \alpha_2) \text{ highest weight module of weight } \lambda$$

The weight λ is called "typical" if $\alpha_1+\alpha_2\neq -1$ and $\alpha_2\neq 0$, otherwise it is called "atypical". Using this, the previous family is split into two types of representations: typical/ atypical if they correspond to typical/atypical highest weight.

Remark 3.1.0.7. *Using the R-matrix, we obtain a braiding on the category of finite dimensional representations of $U_q(sl(2|1))$:*

$$\mathscr{R}_{V,W} := \tau^s \circ (R \curvearrowright V \otimes W)$$

where τ^s is the super flip on two homogeneous elements defined as

$$\tau^s(x\otimes y) = (-1)^{degx\ degy} y\otimes x$$

Proposition 3.1.0.8. *The braiding from the category of representations induces a well defined braid group action, in the following way:*

$$\rho_n : B_n \to Aut_{U_q(sl(2|1))}(V(\lambda)^{\otimes n})$$

$$\rho_n(\sigma_i^{\pm 1}) = Id^{\otimes i-1} \otimes \mathscr{R}^{\pm 1}_{V(\lambda),V(\lambda)} \otimes Id^{n-i-1}$$

Theorem 3.1.0.9. *([28], Lemma 1.3) If $\alpha,\beta\in\mathbb{C}^*, n\in\mathbb{N}$ such that all the modules from the following expression are typical, then $V(0,\alpha)\otimes V(n,\beta)$ is semi-simple and has the following decomposition:*
1) For $n\neq 0$:

$$V(0,\alpha)\otimes V(n,\beta) = V(n,\alpha+\beta)\oplus V(n+1,\alpha+\beta)\oplus V(n-1,\alpha+\beta+1)\oplus V(n,\alpha+\beta+1).$$

2) For $n=0$:

$$V(0,\alpha)\otimes V(0,\beta) = V(0,\alpha+\beta)\oplus V(0,\alpha+\beta+1)\oplus V(1,\alpha+\beta).$$

3.2 The Links-Gould invariant

In this section, we will recall the definition of the Links-Gould invariant for links. After that, in the second part, we will see how this invariant can be recovered from a more general set of renormalized invariants introduced by Geer and Patureau using the representation theory of $U_q(sl(2|1))$.

Let $K := \mathbb{C}(t_0^{\pm\frac{1}{2}}, t_1^{\pm\frac{1}{2}})$ and $V := < v_1, ..., v_4 >$ a K-vector space of dimension 4.

Consider the set $\mathscr{B}$ to be the following ordered basis for V:

$$\mathscr{B} := (v_1 \otimes v_1, ..., v_1 \otimes v_4, v_2 \otimes v_1, ..., v_2 \otimes v_4, ..., v_4 \otimes v_1, ..., v_4 \otimes v_4)$$

Let us denote by $Y := ((t_0 - 1)(1 - t_1))^{\frac{1}{2}}$.

Consider the operator $R \in Aut(V \otimes V)$ given by the following matrix:

$$\begin{bmatrix}
t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & t_0^{\frac{1}{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{\frac{1}{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & t_0^{\frac{1}{2}} & \cdot & \cdot & t_0 - 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0t_1 - 1 & \cdot & \cdot & -t_0^{\frac{1}{2}}t_1^{\frac{1}{2}} & \cdot & \cdot & -t_0^{\frac{1}{2}}t_1^{\frac{1}{2}}Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{\frac{1}{2}} & \cdot & \cdot \\
\cdot & \cdot & t_0^{\frac{1}{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & t_0 - 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{\frac{1}{2}}t_1^{\frac{1}{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{\frac{1}{2}} & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & -t_0^{\frac{1}{2}}t_1^{\frac{1}{2}}Y & \cdot & \cdot & Y & \cdot & \cdot & Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{\frac{1}{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & t_1 - 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{\frac{1}{2}} & \cdot & \cdot & t_1 - 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1
\end{bmatrix}$$

Proposition 3.2.0.1. *[24] The operator $R \curvearrowright V \otimes V$ satisfies the Yang-Baxter equation, and it induces a representation of the braid group:*

$$\varphi_n : B_n \to Aut(V^{\otimes n})$$

$$\varphi(\sigma_i^{\pm 1}) = Id^{i-1} \otimes R^{\pm 1} \otimes Id^{n-i-1}$$

Definition 3.2.0.2. *Consider the operator $\mu \in Aut(V)$ defined as:*

$$\mu = \begin{pmatrix} t_0^{-1} & \cdot & \cdot & \cdot \\ \cdot & -t_1 & \cdot & \cdot \\ \cdot & \cdot & -t_0^{-1} & \cdot \\ \cdot & \cdot & \cdot & t_1 \end{pmatrix}$$

For an endomorphism $f \in End(V)$ which is a scalar times the identity, we denote by $< f >$ the corresponding element of K:

$$f = < f > Id_V$$

Theorem 3.2.0.3. *For any braid $\beta \in B_n$, the following partial trace is a scalar:*

$$Tr_{2,...,n}((Id_V \otimes \mu^{\otimes(n-1)}) \circ \varphi_n(\beta)) \in K \cdot Id_V.$$

The Links-Gould polynomial is defined using this partial trace in the following way:

$$LG(L; t_0, t_1) :=< Tr_{2,...,n}((Id_V \otimes \mu^{n-1}) >$$

Then, this is a well defined link invariant and it has integer coefficients ([40]):

$$LG(L; t_0, t_1) \in \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}].$$

Lemma 3.2.0.4. *In [28], the authors defined a way of constructing modified type invariants using the representation theory of $U_q(sl(2|1))$. Let $n \in \mathbb{N}$, $\alpha_1, ..., \alpha_k \in \mathbb{C}$ and a link L. They color the components of L with the representations $\{V(n, \alpha_1), ..., V(n, \alpha_k)\}$. Then, a modified Reshetikhin-Turaev type construction leads to a well defined link invariant called the Geer-Patureau modified invariant:*

$$(U_q(sl(2|1)), V(n, \alpha_1), ..., V(n, \alpha_k)) \rightarrow F'(L) = f(L) \cdot M^n(L)(q, q^{\alpha_1}, ..., q^{\alpha_k})$$

where f is a function that depends on the linking number of L and

$$M^n(L)(q, q^{\alpha_1}, ..., q^{\alpha_k}) \in \mathbb{Q}(q, q_1, ..., q_k).$$

Theorem 3.2.0.5. *([30],[54]) The modified Geer-Patureau invariants from $U_q(sl(2|1))$ recover the Links-Gould invariant, by a specialisation of coefficients:*

$$LG(L; t_0, t_1)|_{(t_0 = q^{-2\alpha}, t_1 = q^{2\alpha+2})} = \{\alpha\}\{\alpha + 1\} M^0(q, q^\alpha, ..., q^\alpha)$$

3.3 The centralizer algebra LG_n

In this section, we will introduce a sequence of centralizer algebras corresponding to a sequence of tensor powers a fixed representation $V(0, \alpha)$ of the super-quantum group $U_q(sl(2|1))$. The aim is to understand characteristics of this sequence and its relations to the group algebra of the braid group.

Definition 3.3.0.1. *Let us fix $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. The centralizer algebra corresponding to the representation $V(0, \alpha)$ is defined as:*

$$LG_n(\alpha) := End_{U_q(sl(2|1))}\left(V(0, \alpha)^{\otimes n}\right)$$

Then $\{LG_n\}_{n\in\mathbb{N}}$ form a sequence of algebras, each of them included into the next one:

$$LG_{n-1}(\alpha) \subseteq LG_n(\alpha)$$

such that $LG_n(\alpha)$ becomes a bimodule over $LG_{n-1}(\alpha)$.

Remark 3.3.0.2. *From the braid group action defined in Proposition 3.1.0.8, at the level of the braid group algebra it is obtained the following morphism:*

$$\rho_n(\alpha) : \Bbbk B_n \to LG_n(\alpha)$$

Theorem 3.3.0.3. *(Marin-Wagner [67]) The morphism $\rho_n(\alpha)$ is surjective.*

Once we know that this morphism is surjective, it is interesting to study more deeply the image and the kernel of this map. They were studied for small values of n ($n \leq 5$) by Marin and Wagner and there have been a couple of conjectures about them. Firstly, the question would be to compute the dimension of the image. Further on, to study the kernel of the map and which are the relations that are needed to quotient the algebra $\Bbbk B_n$ by, in order to obtain an isomorphism.

Conjecture 5. *(Marin-Wagner [67])*

$$dim(LG_{n+1})(\alpha) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}.$$

Moreover, going further in the study of the morphism $\rho_n(\alpha)$, the following question refers to the difference between the algebra $LG_n(\alpha)$ and $\Bbbk B_n$.

Definition 3.3.0.4. *(The cubic Hecke algebra)*
Let $a, b, c \in \Bbbk^*$. Define the corresponding cubic Hecke algebra as:

$$H_n(a, b, c) := \Bbbk B_n / \left((\sigma_1 - a)(\sigma_1 - b)(\sigma_1 - c)\right)$$

Actually, for specific values of the parameters, the cubic Hecke algebra will lead to the centralizer algebra LG_n.

Proposition 3.3.0.5. *([67]) Let $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ such that $\{1, \alpha, \alpha^2\}$ are linearly independent.*
Consider the parameters $a = -q^{-2\alpha(\alpha+1)}$, $b = q^{-2\alpha^2}$ and $c = q^{-2(\alpha+1)^2}$.

Denote by $H_n(\alpha) = H_n(a,b,c)$. Then the morphism $\rho_n(\alpha)$ factors through the cubic Hecke algebra $H_n(\alpha)$:

$$\begin{tikzcd} \rho_n(\alpha) : \Bbbk B_n \arrow[rr] \arrow[dr] & & LG_n(\alpha) \\ & H_n(\alpha) \arrow[ur, "\tilde{\rho}_n(\alpha)"'] & \end{tikzcd}$$

In [39], Ishii introduced a relation r_2 in $H_3(\alpha)$ and showed that this relation is the kernel of $\tilde{\rho}_3(\alpha)$. However, on the next number of strands, it is still needed to quotient by more relations. In [67], the authors defined a relation $r_3 \in H_4(\alpha)$ and proved that $H_4/(r_2, r_3) \simeq LG_4$. They conjectured that these relations are enough in order to describe the kernel of $\rho_n(\alpha)$ for all values of n.

Definition 3.3.0.6. *Consider the quotient algebra defined by $A_n(\alpha) := H_n(\alpha)/(r_2, r_3)$.*

Conjecture 6. *(Marin-Wagner[67]) For any number of strands $n \in \mathbb{N}$, there is the isomorphism:*

$$A_n(\alpha) \simeq LG_n(\alpha).$$

3.4 Proof of the Conjecture

In this part, we prove the Conjecture 5 by Marin-Wagner. We will use combinatorial tools to control the semi-simple decomposition of the tensor power of $U_q(sl(2|1))$ representations. We encode the semi-simple decomposition of $V(0,\alpha)^n$ as a certain diagram $D(n)$ in the plane. Then, each point will have a weight, which corresponds to a certain multiplicity of a simple representation inside the the n-th tensor power of $V(0,\alpha)$. The final part is to express these weights from the diagram $D(n)$ as a certain ways of counting planar paths of fixed length with prescribed possible moves.

3.4.1 Combinatorial description for the intertwiners of $V(0,\alpha)^{\otimes n}$

As we have seen

$$LG_n(\alpha) = End_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n}).$$

We will describe the intertwiners that occur in the tensor decomposition of $V(0,\alpha)^{\otimes n}$ in a combinatorial way. We are interested in the semi-simple decomposition of $V(0,\alpha)^{\otimes n}$. We remark that since $\alpha \notin \mathbb{Q}$, by an inductive argument it follows that the tensor power $V(0,\alpha)^{\otimes n}$ is semi-simple and the formula 3.1.0.9 can be applied at each step.

Notation 3.4.1.1. *Let us denote the semi-simple decomposition by:*

$$V(0,\alpha)^{\otimes k} = \oplus_{x,y \in \mathbb{N}\times\mathbb{N}} \ (\ T_k(x,y) \otimes V(x, k\alpha + y) \ )$$

where $T_k(x,y)$ is the intertwiner space corresponding to the weight $(x, k\alpha+y)$.

We will codify this decomposition by a graph in the plane with integer coordinates, where each point will have a certain "weight".

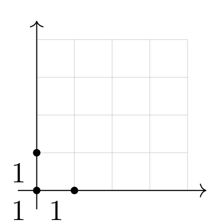
Definition 3.4.1.2. *We say that $D(n)$ is a diagram for $V(0,\alpha)^{\otimes n}$ if it is included in the lattice with integer coordinates and weights natural numbers such that each point $(x,y) \in D(n)$ has the associated multiplicity*

$$t_n(x,y) = dim \ T_n(x,y).$$

This encodes in the position (x,y) the multiplicity of the module with highest weight that moves from the fundamental weight $(0, n\cdot\alpha)$ with x from 0 and with y from $n\alpha$.
In other words, we can think that the origin of diagram $D(n)$ has the coordinates $(0, n\cdot\alpha)$.

As we can see, we can deduce the tensor decomposition of $V(0,\alpha)^{\otimes n}$ by reading the non-zero multiplicities associated to points from $D(n)$. Let us see some examples:

n = 2 $\quad V(0,\alpha)\otimes V(0,\alpha) = V(0,2\alpha)\oplus V(0,2\alpha+1)\oplus V(1,2\alpha).$



(3.2)

Case $\mathbf{n=3}$

$$V(0,\alpha)^{\otimes 3} = (V(0,2\alpha)\oplus V(0,2\alpha+1)\oplus V(1,2\alpha))\otimes V(0,\alpha) =$$

$$= (V(0,2\alpha)\otimes V(0,\alpha))\oplus(V(0,2\alpha+1)\otimes V(0,\alpha))\oplus(V(1,2\alpha)\otimes V(0,\alpha)) =$$

$$= (V(0,3\alpha)\oplus V(0,3\alpha+1)\oplus V(1,3\alpha))\oplus$$

$$\oplus(V(0,3\alpha+1)\oplus V(0,3\alpha+2)\oplus V(1,3\alpha+1))\oplus$$

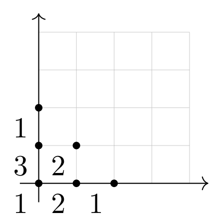
$$\oplus(V(1,3\alpha)\oplus V(1,3\alpha+1)\oplus V(0,3\alpha+1)\oplus V(2,3\alpha)).$$

We conclude that the semi-simple decomposition of the third tensor power is the following:

$$V(0,\alpha)^{\otimes 3} = V(0,3\alpha)\oplus 3\cdot V(0,3\alpha+1)\oplus V(0,3\alpha+2)$$

$$\oplus 2\cdot V(1,3\alpha)\oplus 2\cdot V(1,3\alpha+1)\oplus V(2,3\alpha).$$

We obtain the diagram for $D(3)$ in the following way:



(3.3)

Case $\mathbf{n=4}$

$$V(0,\alpha)^{\otimes 4} = V(0,\alpha)^{\otimes 3}\otimes V(0,\alpha) =$$

$$= ((V(0,3\alpha)\otimes V(0,\alpha))\oplus 3\cdot(V(0,3\alpha+1)\otimes V(0,\alpha))\oplus(V(0,3\alpha+2)\otimes V(0,\alpha))\oplus$$

$$\oplus 2\cdot(V(1,3\alpha)\otimes V(0,\alpha))\oplus 2\cdot(V(1,3\alpha+1)\otimes V(0,\alpha))\oplus(V(2,3\alpha)\otimes V(0,\alpha)) =$$

$$= (V(0,4\alpha)\oplus V(0,4\alpha+1)\oplus V(1,4\alpha))\oplus$$

$$\oplus(3V(0,4\alpha+1)\oplus 3V(0,4\alpha+2)\oplus 3V(1,4\alpha+1))\oplus$$

$$\oplus(V(0,4\alpha+2)\oplus V(0,4\alpha+3)\oplus V(1,4\alpha+2))\oplus$$

$$\oplus(2V(1,4\alpha)\oplus 2V(1,4\alpha+1)\oplus 2V(0,4\alpha+1)\oplus 2V(2,4\alpha))\oplus$$

$$\oplus(2V(1,4\alpha+1)\oplus 2V(1,4\alpha+2)\oplus 2V(0,4\alpha+2)\oplus 2V(2,4\alpha+1))\oplus$$

$$\oplus(V(2,4\alpha)\oplus V(2,4\alpha+1)\oplus V(1,4\alpha+1)\oplus V(3,4\alpha)).$$

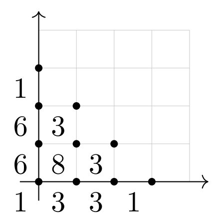
We conclude that the semi-simple decomposition of the fourth tensor power is the following:

$$V(0,\alpha)^{\otimes 4} = V(0,4\alpha) \oplus 6 \cdot V(0,4\alpha+1) \oplus 6 \cdot V(0,4\alpha+2) \oplus$$

$$\oplus V(0,4\alpha+3) \oplus 3 \cdot V(1,4\alpha) \oplus 8 \cdot V(1,4\alpha+1) \oplus$$

$$\oplus 3 \cdot V(1,4\alpha+2) \oplus 3 \cdot V(2,4\alpha) \oplus 3 \cdot V(2,4\alpha+1) \oplus V(3,4\alpha).$$

We obtain diagram $D(4)$ with the following weights:



(3.4)

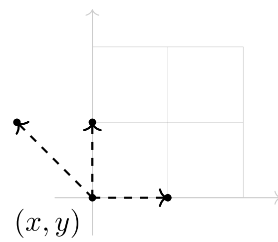
In the sequel, we will describe how the diagrams $D(n)$, can be constructed in a recursive way. More precisely, if we suppose that we know the diagram $D(n)$, then by applying certain moves, we will be able to obtain $D(n+1)$. Let us start with $V(m,\beta)$. We will encode the decomposition of $V(m,\beta) \otimes V(0,\alpha)$ in a lattice. Let us think that initially, $V(m,\beta)$ is encoded by diagram D which has as origin (m,β) and the corresponding multiplicity 1.

Definition 3.4.1.3. **a**) *From the Theorem 3.1.0.9, for any $x \in \mathbb{N} \setminus \{0\}$ and any $y \in \mathbb{N}, n \in \mathbb{N}$ we have the decomposition:*

$$V(0,\alpha) \otimes V(x,n\alpha+y) = V(x,(n+1)\alpha+y) \oplus V(x+1,(n+1)\alpha+y) \oplus$$

$$\oplus V(x-1,(n+1)\alpha+y+1) \oplus V(x,(n+1)\alpha+y+1).$$

We call the effect of tensoring $V(x,n\alpha+y)$ with $V(0,\alpha)$ a "blow up of type (x,y)" and $B(x,y)$ the new corresponding diagram:



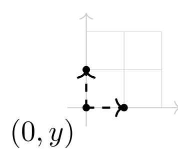
(3.5)

b) *If the first coordinate $x = 0$, then for any $y \in \mathbb{N}$ and $n \in \mathbb{N}$, the decomposition is as follows:*

$$V(0,\alpha) \otimes V(0, n\alpha + y) = V(0,(n+1)\alpha + y) \oplus V(1,(n+1)\alpha + y) \oplus$$

$$\oplus V(0,(n+1)\alpha + y + 1).$$

We call the effect of tensoring $V(0, n\alpha + x)$ with $V(0,\alpha)$ a "blow up of type $(0, y)$" and $B(0, y)$ the new corresponding diagram.



(3.6)

Lemma 3.4.1.4. *The diagram $D(n + 1)$ can be obtained from $D(n)$, by blowing up each point $(x, y) \in D(n)$ with $B(x, y)$ for $t_n(x, y)$ times and add in each vertex all the new multiplicities.*

Proof. Suppose we have $D(n)$. This means that:

$$V(0,\alpha)^{\otimes n} = \oplus_{x,y \in \mathbb{N}\times\mathbb{N}} \ (t_n(x, y) \cdot V(x, n\alpha + y))$$

In order to obtain the multiplicities that occur in $D(n + 1)$, we have:

$$V(0,\alpha)^{\otimes n+1} = \oplus_{x,y \in \mathbb{N}\times\mathbb{N}} \ (t_n(x, y) \cdot (V(x, n\alpha + y) \otimes V(0,\alpha)) \ ) \quad (*)$$

On the other hand, the multiplicities t_{n+1} occur in the following way:

$$V(0,\alpha)^{\otimes n+1} = \oplus_{x,y \in \mathbb{N}\times\mathbb{N}} \ (t_{n+1}(x, y) \cdot V(x,(n+1)\alpha + y) \ )$$

Using the previous formula $(*)$, we notice that the diagram $D(n+1)$, has some extra weights with respect to the diagram $D(n)$. More precisely, each term $(V(x, n\alpha+y) \otimes V(0, \alpha))$ will add to the weights from $D(n)$, some extra multiplicities corresponding to a blow up of center

$$(x+0, (n\alpha+y)+\alpha) = (x, (n+1)\alpha + y).$$

This is encoded in $D(n+1)$ as the blow-up $B(x, y)$ with center (x, y). Counting the multiplicities, for each point (x, y), we'll have to do the blow-up $B(x, y)$ for $t_n(x, y)$ times. In this way, we obtain $t_{n+1}(x, y)$. □

Up to this point, we saw how to construct the recursive relation that relates $D(n)$ and $D(n+1)$. However, this is still at the theoretical level. In the following part, we will use the fact that we know the initial step corresponding to the diagram $D(1)$, and using the recursive relation we will show how each multiplicity $t_n(x, y)$ can be described in a natural way, using a method of counting paths in the plane.

Remark 3.4.1.5. *1)In $D(n+1)$, for each point (x, y), the total multiplicity if obtained by adding all the multiplicities of the points from $D(n)$, which can arrive to (x, y) using one of the following moves:*

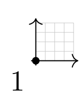
M1)	*stay move*	$(x, y) \to (x, y)$
M2)	$\longrightarrow$	$(x, y) \to (x+1, y)$
M3)	$\uparrow$	$(x, y) \to (x, y+1)$
M4)	$\nwarrow$	$(x, y) \to (x-1, y+1)$ *if* $x > 0$.

Here, the reason for having the condition that the move M_4 can be done just if $x > 0$, is the fact that a coefficient that decreases the y coordinate occurs in the blow-up $B(x, y)$ if and only if $x > 0$.

2)If we start from $D(n-1)$, we can obtain $D(n+1)$, by counting certain paths of length 2 in the integer lattice (with the corresponding multiplicities as in $D(n-1)$. In this way, applying twice the first remark we conclude that:

$$t_{n+1}(x, y) = \textit{card}\ \{\ \textit{paths of length 2 starting from points in } D(n-1) \textit{ which}$$
$$\textit{end in } (x, y),\ \textit{with the possible moves } M_1, M_2, M_3 \textit{ or } M_4\ \}.$$

3) We will iterate this argument by induction, using as initial data the diagram $D(1)$ which has the following form:



(3.7)

We obtain the following combinatorial description for the intertwiners spaces:

Theorem 3.4.1.6. *In $D(n+1)$, for each point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ the associated multiplicity has the formula:*

$t_{n+1}(x, y) =$ *number of paths in the plane from* $(0, 0)$ *to* (x, y) *of length* (n)

and possible moves M1, M2, M3 or M4 with the condition that

they do not have any point with a negative coordinate on the x-axis.

Remark 3.4.1.7. *In the diagram $D(n + 1)$, the only points that have non-zero associated weights are inside the standard simplex $\Delta_n \in \mathbb{R}^2$ (the length of the edge of Δ_n is n).*

Notation 3.4.1.8. *Consider the following set of paths:*

$$P_{n+1}(x, y) := \{\textit{planar paths from } (0, 0) \textit{ to } (x, y) \textit{ of length } n$$
$$\textit{with possible moves M1, M2, M3 or M4}$$
$$\textit{with the condition that}$$
$$\textit{they are contained in the positive cadran}\}$$

Remark 3.4.1.9. *For any number $n \in \mathbb{N}$ and $x, y \in \Delta_n$:*

$$t_{n+1}(x, y) = |P_{n+1}(x, y)|$$

Now we will see which is the relation between the dimension of centralizer algebra $LG_n(\alpha)$ and the dimensions of its intertwiner spaces.

Remark 3.4.1.10. *As we have seen, there is the following decomposition of the tensor power of the 4-dimensional representation:*

$$V(0, \alpha)^{\otimes n} = \oplus_{x,y \in \mathbb{N}\times\mathbb{N}} \ (t_n(x, y)V(x, n\alpha + y)\ )$$

where $t_n(x, y)$ is the cardinality of the intertwiner space corresponding to the weight $(x, n \cdot \alpha + y)$.

Proposition 3.4.1.11. *From [28], for typical $(n, \alpha) \in \Lambda$, $V(n, \alpha)$ is simple representation and moreover:*

$$Hom_{U_q(sl(2|1))} (V(n, \alpha), V(m, \beta)) \simeq \delta^{(m,\beta)}_{(n,\alpha)} \cdot \Bbbk Id.$$

In our case, using that $\alpha \notin \mathbb{Q}$, we deduce that for any $(x,y) \in \Delta_n$, all modules $V(x, n\alpha + y)$ are typical. This shows the following decomposition of the endomorphism ring of the tensor product:

$$End_{U_q(sl(2|1))}\left(V(0,\alpha)^{\otimes n}\right) \simeq \oplus\ End_{U_q(sl(2|1))}\left(t_n(x,y)\ V(x, n\alpha + y)\right)\right) \simeq$$

$$\simeq^{3.4.1.11} \oplus M(t_n(x,y), \mathbb{k}).$$

Corollary 3.4.1.12. *From the previous decomposition of LG_n using isotipic components, we obtain a formula for its dimension:*

$$dim\ LG_n(\alpha) = \sum_{x,y \in \mathbb{N}\times\mathbb{N}}^{x+y\leq n-1} t_n(x,y)^2.$$

Corollary 3.4.1.13. *From this formula of $LG_n(\alpha)$, we conclude that:*

$$dim\ LG_{n+1}(\alpha) = \sum_{x,y \in \mathbb{N}\times\mathbb{N}}^{x+y\leq n} |P_{n+1}(x,y)|^2.$$

3.4.2 Computation for the dimension of $LG_{n+1}(\alpha)$

In this section we will finish the proof of Conjecture 5. In [67], it is mentioned that F. Chapoton remarked that the conjectured dimension of LG_{n+1} coincides with a combinatorial formula for a way of counting pairs of paths in the plane.

Theorem 3.4.2.1. *[4], [80], There is the following identity:*

$$\frac{(2n)!(2n+1)!}{(n!(n+1)!)^2} = \text{number of pairs of disjoint paths in the}$$

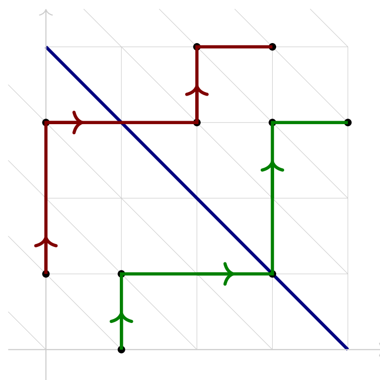
$$(n+1)\times(n+1) \text{ square, which go either}$$

$$\text{upwards or straight to the right} \uparrow \text{or} \rightarrow$$

$$\text{between } (0,1)\rightarrow(n,n+1) \text{ and } (1,0)\rightarrow(n+1,n).$$

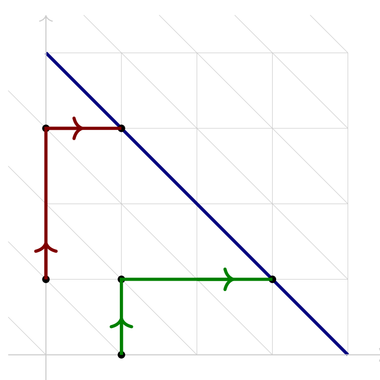
Definition 3.4.2.2. *We will denote by C_{n+1} this set of pairs of paths.*

We will prove the Conjecture using the previous formula 3.4.1.13 for $dim\ LG_{n+1}(\alpha)$ and this result for the conjectured number.
Since in the formula for the dimension of $LG_{n+1}(\alpha)$, there are counted all multiplicities $t_{n+1}(x, y)$, for $(x, y) \in \Delta_n$, we will describe C_{n+1} as an union of subsets, indexed by the same set Δ_n. Having a pair of paths in the square, the idea is to remember where those paths "cut the principal diagonal", and use this data as an indexing set.



(3.8)

$C_{n+1}(a, b)$ $(n + 1 - a, a)$ $(n + 1 - b, b)$



(3.9)

$C^{\Delta}_{n+1}(a, b)$ $(n + 1 - a, a)$ $(n + 1 - b, b)$

Definition 3.4.2.3. *Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ such that $a, b \leq n+1$ and $a > b$. We consider the following sets:*

$$C_{n+1}(a,b) := \{ \textit{ pairs of paths in } C_{n+1} \textit{ that cut the principal diagonal of the square precisely in the points } (n+1-a, a) \textit{ and } (n+1-b, b) \}.$$

$$C^{\Delta}_{n+1}(a,b) := \{ \textit{ pairs of disjoint paths contained in the simplex } \Delta_{n+1} \textit{ between the points } (1,0) \textit{ and } (n+1-a, a) \textit{ respectively } (0,1) \textit{ and } (n+1-b, b) \}.$$

Remark 3.4.2.4. *Using these notations, C_{n+1} can be expressed in the following way:*

$$C_{n+1} = \bigcup_{a,b \leq n+1}^{a>b} C_{n+1}(a,b)$$

$$C_{n+1}(a,b) \simeq C^{\Delta}_{n+1}(a,b) \times C^{\Delta}_{n+1}(a,b)$$

The second bijection can be established by starting with a path from $C_{n+1}(a, b)$, and cutting it along the principal diagonal of the square. In this way, there are obtained two paths in $C^{\Delta}_{n+1}(a, b)$.

Proposition 3.4.2.5. *From the previous remarks and definitions we conclude that:*

$$C_{n+1} = \bigcup_{a,b \leq n+1}^{a>b} \left(C^{\Delta}_{n+1}(a,b) \times C^{\Delta}_{n+1}(a,b) \right)$$

$$\mid C_{n+1} \mid = \sum_{a,b \leq n+1}^{a>b} \mid C^{\Delta}_{n+1}(a,b) \mid^2$$

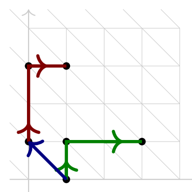
Notation 3.4.2.6. *For $(a, b) \in \mathbb{N} \times \mathbb{N}$ with $a, b \leq n$ and $a \geq b$, denote by*

$$D^{\Delta}_{n}(a,b) := \{ \textit{ pairs of paths in } \Delta_n \textit{ between the points } (0,0) \to (n+1-a, a) \textit{ and } (0,0) \to (n+1-b, b) \textit{ that can intersect each other just in integer points, but they do not cross each other } \}.$$

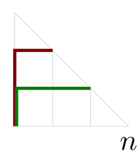
Proposition 3.4.2.7. *There is a bijection between the following sets:*

$$C_{n+1}^{\Delta}(a,b) \simeq D_n^{\Delta}(a,b-1)$$

Proof. Let $C, D \in C_{n+1}^{\Delta}(a,b)$ to be a pair of paths.
By modifying $C \to C + (-1,1)$, we will obtain $C + (-1,1), D \in D_n^{\Delta}(a,b-1)$ (where here the simplex Δ_n is seen as being bounded by the points $(0,1)$, $(n,1)$, and $(n+1,0)$).
After that, it can be easily shown that this function is a bijection. □



(3.10)



(3.11)

Lemma 3.4.2.8. *From the last remark, it can be deduced that the cardinality of C_n can be expressed using pairs of paths in the simplex Δ_n:*

$$\mid C_{n+1} \mid = \sum_{a,b \leq n}^{a \geq b} \mid D_n^{\Delta}(a,b) \mid^2$$

Remark 3.4.2.9. *For any pair of paths*

$$C_1 = ((C_1)_x^k, (C_1)_y^k) \text{ and } C_2 = ((C_2)_x^k, (C_2)_y^k) \text{ in } D_n^{\Delta}(a,b)$$

the condition that they do not cross each other can be stated as:

$$(C_1)_y^k \leq (C_2)_y^k, \quad \forall k.$$

Now, we arrive at the last part, and we will show a correspondence between $P_{n+1}(x,y)$ and $D_n^{\Delta}(a,b)$.

Lemma 3.4.2.10. *We have the following correspondence between the previous two sets of planar paths:*

$$P_{n+1}(x_0, y_0) \simeq D_n^{\Delta}(x_0 + y_0, y_0),$$

for any fixed point in the length n simplex $(x_0, y_0) \in \Delta_n$.

Proof. Let $C_1, C_2 \in D_n^{\Delta}(x + y, y)$. This pair of paths can be encoded in a sequence of moves of four types. We start at the common point $(0, 0)$ which correspond to the step 0. Then, we see how each of the two paths changes from one step to the other.
Suppose that $(x_1, y_1) \in C_1$ and $(x_2, y_2) \in C_2$ which correspond to the k^{th} step. In order to pass to the $(k + 1)^{st}$ step, we have four possibilities.

Movements corresponding to $D_n^{\Delta}(x + y, y)$
The pair $((x_1, y_1),\ (x_2, y_2))$ for which we know the condition $y_2 \geq y_1$, can be modified in order to arrive at the next step, by adding one out of the four following pairs of vectors:

$$((0, 1),\ (0, 1))$$

$$((1, 0),\ (1, 0))$$

$$((1, 0),\ (0, 1))$$

$$((0, 1),\ (1, 0))$$

On the other hand, for any path $C \in P_{n+1}(x, y)$, it can be also encoded by specifying which move we do from the k^{th} step to the $(k + 1)^{st}$step.

Movements corresponding to $P_{n+1}(x, y)$
Let (x, y) a point in C and we know the condition that $x \geq 0$. In order to pass to the next point in the path, we can modify the point with one out of the following vectors:

$$(0, 0)$$

$$(0, 1)$$

$$(1, 0)$$

$$(-1, 1)$$

Now, we want to establish a correspondence between the two types of movements. We will define a function

$$f : D_n^{\Delta}(x_0 + y_0, y_0) \to P_{n+1}(x_0, y_0).$$

Let $C_1, C_2 \in D_n^{\Delta}(x+y, y)$. We want to send each pair of points

$$((x_1, y_1) \in C_1, (x_2, y_2) \in C_2) \text{ into } f\left((x_1, y_1), (x_2, y_2)\right)$$

such that it satisfies the restrictions from $P_{n+1}(x_0, y_0)$.

Since we know the condition $y_2 \geq y_1$, it would be natural to send

$$f((x_1, y_1), (x_2, y_2))_1 = y_2 - y_1$$

which would ensure us the necessary condition for positivity.

Consider f to be defined by the following formula:

$$f\left((x_1, y_1), (x_2, y_2)\right) = (y_2 - y_1, x_2).$$

Then $f((0,0),(0,0)) = (0,0)$, so it preserves the initial points. Now we can check that this transformation, preserves correspondingly all the possible movements that we described in the two cases, in the following way:

$$\begin{array}{ccc}
((x_1, y_1),\ (x_2, y_2)) & \longrightarrow & (y_2 - y_1, x_2) \\
(0,1),\ (0,1) & \longleftrightarrow & (0,0) \\
(1,0),\ (1,0) & \longleftrightarrow & (0,1) \\
(1,0),\ (0,1) & \longleftrightarrow & (1,0) \\
(0,1),\ (1,0) & \longleftrightarrow & (-1,1)
\end{array}$$

This shows that the function f is a well-defined bijection between

$$D_n^{\Delta}(x_0 + y_0, y_0) \text{ and } P_{n+1}(x_0, y_0).$$

$\square$

Using the previous combinatorial interpretations, we obtain the result:

Theorem 3.4.2.11.

$$dim(LG_{n+1}(\alpha)) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}.$$

Proof. Using the formulas from Corollary 3.4.1.13, Theorem 3.4.2.1, Lemma 3.4.2.8 and Lemma 3.4.2.10, we obtain the proof of the Marin-Wagner Conjecture. $\square$

Chapter 4

Further directions

In recent years, the study of categorifications of link invariants provided a powerful tool that has led to many important results in knot theory. Khovanov homology is categorification for the original Jones polynomial. It was proved that this categorification detects the unknot, whereas the question whether the Jones polynomial itself detects the unknot is still open. On the other hand, Ozsváth and Szabó ([71], [72]) and Rasmussen ([76]) defined a categorification of the Alexander polynomial using Heegaard Floer homology. In ([75]), Rasmussen introduced an invariant using Khovanov homology and showed that it gives a bound for the slice genus which lead to a proof of the Milnor conjecture.

The main problems and questions that I would like to pursue to study are related to the understanding at a deep level of the connections and relations between the initial algebraic description of quantum invariants on one side and topological constructions or topological applications on the other side. This comes from the main theme and aim of my thesis. More precisely, I am interested to study categorifications for quantum invariants that have a geometric nature, using Floer type methods. In view of the program 0.5, the path that I am interested to follow is to start with a quantum invariant, and once we find a topological model as a graded intersection pairing, this will provide a tool to study possible geometrical categorifications. The first immediate invariants that I am interested to study are the coloured Jones polynomials and coloured Alexander polynomials

Coloured Jones and Coloured Alexander polynomials

The Alexander polynomial and the Jones polynomial for a links are invariants which are defined initially in different manners, but both of them can be seen as quantum invariants. Their definitions come both from the representation theory of the quantum group $U_q(sl(2))$. As we have seen, the coloured Jones polynomials come from the representation theory of $U_q(sl(2))$ with generic q and the first term of this sequence is the original Jones polynomial. On the other hand, in 1992, Akustu, Deguchi and Ohtsuki([5]) introduced a sequence of quantum invariants for links called the coloured Alexander polynomials, using the representation theory of $U_\xi(sl(2))$ (where ξ is a root of unity). They were the first renormalized-type quantum invariants for knots. In the sequel we will present the similarities and differences between the two type of constructions.

The construction of renormalized type invariants is related to the problem that passing from q generic to q root of unity, the so called quantum dimensions of representations are generically zero. Moreover, the classical Reshetikhin-Turaev construction encodes those quantum dimensions, leading to invariants that vanish. The idea to overcome the vanishing of the Reshetikhin-Turaev functor, is to cut one strand of the link, and apply the functor to the $(1,1)$-tangle obtained in this way.

Comparison between the two quantum invariants

The representation theory of $U_q(sl(2))$ changes totally if we pass from q generic to q root of unity. If $\xi^{2N}=1$ then the simple representations of $U_\xi(sl(2))$ form a continuous family $\{V_\lambda\}_{\lambda\in\mathbb{C}}$. Here, for constructing link invariants, we fix q, but the parameter is given by the highest weight of the representation V_λ.

The quantum groups $U_q(sl(2))$ and $U_\xi(sl(2))$ are braided, and using their representation theories there are two families of braid group representations:

$$\varphi_n^N : B_n \to Aut(V_N^{\otimes n}) \qquad\qquad \psi_n^\lambda : B_n \to Aut(V_\lambda^{\otimes n})$$

$$\forall N \in \mathbb{N}^* \qquad\qquad\qquad \forall \lambda \in \mathbb{C}$$

From the algebraic structure of those quantum groups, there is a quantum trace on the category of representations (which means closing with all cups and caps) and a partial quantum trace (which corresponds to cut one strand

and close all the others).

$$qtr_n : End(V_N^{\otimes n}) \to \mathbb{Z}[q^{\pm 1}] \qquad pqtr_n : End(V_\lambda^{\otimes n}) \to End(V_\lambda) \simeq \mathbb{C}$$

In other words, at the level of braid group representation, if we start with a link L and write $L = \hat{\beta}$, the Nth coloured Alexander invariant invariant is defined

$$\phi_N(L,\lambda) \approx pqtr_{2n}(\psi_{2n}^{\lambda}(\beta \cup I_n))$$

On the other hand, the coloured Jones polynomial can be defined as:

$$J_N(L,q) = qtr_{2n}(\varphi_{2n}^{N}(\beta \cup I_n))$$

We remark the fact that actually, the ADO polynomial and the coloured Jones polynomial, are constructed as being mirror to one another.

(q generic, V_N fixed)	$\big(\xi = 2N^{th}$ root of 1, $V_\lambda$variable$\big)$
total quantum trace	partial quantum trace
$(U_q(sl(2)), V_N) \to J_N(L,q)$	$(U_\xi(sl(2)), V_\lambda) \to \phi_N(L,\lambda)$
Coloured Jones polynomial	Coloured Alexander polynomial

Towards Geometrical Categorifications

A research direction that I would like to pursue concerns topological categorifications for certain quantum invariants, following the second question (0.5) of the research program that I started in this thesis. For the case of the coloured Jones polynomials there are known categorifications constructed with algebraic tools, but there is no geometrical categorification known. A further plan is to continue the study of the homological model that we constructed for the coloured Jones polynomials and investigate if the Floer theory coming from that leads to a well defined invariant. We discuss details about it in 4 and present a precise conjecture (7) concerning this question.

Concerning the Alexander polynomials, Bigelow, Cattabriga and Florens ([19]) used the Lawrence representations, in order to extend the original Alexander polynomial to tangles. Further more, they obtained from this model (and later Kalinin by a direct method) a homological interpretation for the original Alexander polynomial.

For the case of coloured Alexander polynomials there are no known topological interpretations yet. Moreover, there are not known any type of categorifications for these invariants. Firstly we conjecture that it is possible to find a topological model for coloured Alexander polynomials 8 and discuss details about that in 4. Pursuing this line, if we have a homological model for the coloured Alexander invariants, we can go further and study a Floer type categorification that arises from this description, which is presented in Conjecture 9.

A summary of these questions and models is presented below.

Algebraic construction-Reshetikhin-Turaev method

$(U_q(sl(2)), V_N)$	$\rightarrow$	Coloured Jones polynomial		**Jones polynomial**
(q generic, $N \in \mathbb{N}^*$)		$J_N(L,q)$	$\dashrightarrow^{N=2}$	$J_2(L,q)$
Description		Quantum inv / No skein		Quantum inv/ Skein
		$\downarrow$		$\downarrow$
Homological model		Theorem 1.7.0.1		Bigelow-Lawrence(2001)
Geometrical		???		Symplectic Khovanov
Categorification		Conjecture 7		Homology
$(U_q(sl(2)), V_\lambda)$	$\rightarrow$	Coloured Alexander polynomial		**Alexander polynomial**
$(q^{2N}=1,\ \lambda \in \mathbb{C})$		$\Phi_N(L,\lambda)$	$\dashrightarrow^{N=2}$	$\Delta(L,t)$
Description		Quantum inv / No skein		Quantum inv / Skein
		$\downarrow$		$\downarrow$
Homological model		Conjecture 8		Bigelow-Kalinin
Geometrical		???		Heegaard Floer
Categorification		Conjecture 9		Homology

Geometrical interpretation-Homological braid group representations

Floer Categorification for coloured Jones polynomials

This project is the second part of the program that I started in my PhD. In Theorem 1.7.0.1, we described a topological model for the coloured Jones polynomials. The question is to study the Floer type homology coming from this model and whether this leads to a categorification for these invariants.

Conjecture 7. *(***Categorification for the coloured Jones polynomials***) The graded Floer homology groups $HF^N_m(\beta_{2n})$ coming from the homological model from Theorem 1.7.0.1 define link invariants and give a geometrical categorification for the coloured Jones polynomial $J_N(L,q)$.*

Global Strategy We will start with the punctured disk in the plane $\mathbb{D}_n = \mathbb{D}^2 \setminus \{p_1,...,p_n\}$ and consider the unordered configuration space in it $C_{n,m} = Conf_m(\mathbb{D}_n)$. Then let $\tilde{C}_{n,m}$ be the covering corresponding to the local system from Section 1.3.

Let us fix a colour $N \in \mathbb{N}$ which corresponds to the coloured Jones invariant that we want to study. In Theorem 1.7.0.1, we have constructed two families of homology classes which leave into the homology this covering:

$$\{\mathscr{F}^N_n \in H_{2n,n(N-1)}|_{\alpha_{N-1}}\}_{n\in\mathbb{N}} \qquad \{\mathscr{G}^N_n \in H^{\partial}_{2n,n(N-1)}|_{\alpha_{N-1}}\}_{n\in\mathbb{N}}.$$

Further on we will discuss about the link that we want to study.

Let L be link and $\beta_{2n} \in B_{2n}$ such that $L = \hat{\hat{\beta}}^{or}_{2n}$. We have proved that there is the following expression:

$$J_N(L,q) =< \beta_{2n}\mathscr{F}^N_n, \mathscr{G}^N_n > |_{\alpha_{N-1}}$$

The Lawrence representation $H_{n,m}$ and its dual $H^{\partial}_{n,m}$ are generated by homology classes of m-dimensional Lagrangian submanifolds in $\tilde{C}_{n,m}$ called "multiforks" (1.3.2.2) and "barcodes" (1.4.1.2). Both are lifts of Lagrangian submanifolds from $C_{n,m}$. This means that $(\beta_{2n})\mathcal{F}_n$, $\mathcal{G}_n$ are homology classes given by a combination of lifts of Lagrangian submanifolds in $C_{n,m}$.

In 1.4.2, we saw how the Blanchfield pairing $<,>$ which is defined between homologies of the covering slace $\tilde{C}_{n,m}$, can be computed using a graded intersection in the base space $C_{n,m}$, the graduation coming from the local system. From this we conclude that $< (\beta_{2n}F^N_n, \mathscr{G}^N_n >$ is a linear combination of graded intersections between Lagrangian submanifolds in $C_{2n,n(N-1)}$.

Our plan is to apply graded Floer homology to each graded intersection from before, and define the Floer homology groups $HF^N_m(\beta_{2n})$. The further question would be whether the construction of the Floer groups $HF^N_m(\beta_{2n})$ are invariant with respect to the Markov moves.

Topological model for the coloured Alexander invariants

This project concerns the research program presented in the introduction (0.5), for the case of coloured Alexander polynomials. The precise aim is to give a topological interpretation for the coloured Alexander polynomials. Further on, our aim is to study the graded Floer homology groups coming from this model.

Conjecture 8. *(***Topological model for coloured Alexander invariants***) Let $N \in \mathbb{N}$ to be the colour of the invariant. Consider $n \in \mathbb{N}$.*

Then there exist two families of homology classes

$$\mathscr{E}_n^N \in H_{2n-1,(n-1)(N-1)} \qquad \mathscr{D}_n^N \in H^{\partial}_{2n-1,(n-1)(N-1)}$$

such that if L is a link and $\beta \in B_n$ with $L = \hat{\beta}$ (normal closure), the N^{th} coloured Alexander invariant has the formula:

$$\phi_N(L,\lambda) =< (\beta_n \cup I^{n-1})\mathscr{E}_n^N, \mathscr{D}_n^N > |_{\psi_\lambda}$$

Global strategy The main idea for this model is the fact that the Reshetikhin-Turaev construction evaluated on a link arrives naturally in a particular highest weight space. The first subtlety concerns the fact that the Coloured Alexander invariant in a renormalized invariant for links, which is reflected through a partial quantum trace type construction. Secondly, opposite to the case for $U_q(sl(2))$ with generic q, for roots of unity the representations are not self-dual. In the sequel we sketch a plan concerning this topological model.

Firstly we fix a natural number $N \in \mathbb{N}$. We will work with $U_\xi(sl(2))$ with $\xi^{2N} = 1$ and its representations $\{V_\lambda | \lambda \in \mathbb{C}\}$.

Let us consider L to be a link and $\beta_n \in B_n$ such that $L = \hat{\beta}_n$ (normal closure). By cutting the first strand of this closure, we get a $(1,1)$ tangle which is obtained from $\beta \cup I^{n-1}$ by joining the strands $2,...,n$ with the corresponding ones from I^{n-1} with caps and cups. We will study the Reshetikhin-Turaev functor at these 3 levels of the tangle.

We will start with $v_0 \in V_\lambda$. The coloured Alexander polynomial $\phi_N(L,\lambda)$ will be the coefficient of v_0 that is obtained after applying the functor.

1) Following the functor at the bottom level (corresponding to the cups), we will arrive in the tensor power of V_λ and V_λ^*. The first remark is that we arrive actually in a particular highest weight space:

$$W^\lambda_{2n-1,(n-1)(N-1)} \subseteq V_\lambda^{\otimes n} \otimes V_\lambda^{*\otimes n-1}$$

2) We remark that the Kohno's Theorem that gives a homological counterpart for highest weight spaces in the Verma module works at this root of unity, following the discussion from 1.5.2.4. Here there is an issue coming from the fact that we have highest weight spaces inside tensor power of modules with different weights.

We conjecture the existence of a Lawrence type representation $H_{2n-1,(n-1)(N-1)}$, defined using the configuration space of $(n-1)(N-1)$ points in the $2n-1$ punctured disc with a choice of a certain local system such that, after an identification of the parameters ψ_λ (which depends on λ) we have the following isomorphism:

$$B_{2n-1} \curvearrowright W^{\lambda}_{2n-1,(n-1)(N-1)} \simeq H_{2n-1,(n-1)(N-1)}|\psi_\lambda \curvearrowleft B_{2n-1}$$

After that, we can show that the invariant can be obtained considering all the construction through this highest weight space.

3) We will consider a dual Lawrence representation $H^{\partial}_{2n-1,(n-1)(N-1)}$ and study the non-degeneracy of the pairing $<,>$ specialised at roots of unity:

$$<,> |_{\psi_\lambda} : H_{2n-1,(n-1)(N-1)}|_{\psi_\lambda} \otimes H^{\partial}_{2n-1,(n-1)(N-1)}|_{\psi_\lambda} \to \mathbb{C}$$

4) Once we have a non-degenerate pairing, the homological model will be obtained with a machinery that constructs the homological counterparts of caps and cups:

$$\mathscr{E}^N_n \in H_{2n-1,n(N-1)} \text{ and } \mathscr{D}^N_n \in H^{\partial}_{2n-1,n(N-1)}.$$

We would like to emphasise the role of the partial quantum trace in this construction (in comparison to a total quantum trace). We can see directly on the homological model, that the effect of the partial trace appears in the Lawrence type representation, as we work with configuration space of points in the disk where we remove one more puncture in comparison to the one used for the total trace. This is the difference between the model for the coloured Jones polynomial and the model for the coloured Alexander invariant. Further on, such a model will be suitable for a lagrangian Floer Homology type construction.

Conjecture 9. *(***Categorification for coloured Alexander invariants***)* *The graded Floer homology groups $HF^N_m(\beta_n)$ coming from the homological model from Conjecture 8, are invariant to Markov moves and lead to a categorification for the coloured Alexander polynomial $\phi_N(L,\lambda)$.*

Bridges towards known categorifications

A guiding line and direction that we also have in mind is oriented towards the categorifications that are already known. Concerning the coloured Jones polynomials, Khovanov ([51]) and Beliakova and Wehrli ([13]) defined categorifications for these invariants using diagrammatic and combinatorial techniques.

Moreover, for the original Jones polynomial it is known a geometrical type categorification. The results of Khovanov and Seidel ([52]), Seidel and Thomas ([79]) and Seidel and Smith ([78]) lead to a homology theory coming from geometry of nilpotent slices called symplectic Khovanov homology. In 2015, Abouzaid and Smith ([2], [3]) proved that the symplectic Khovanov homology coincides with the combinatorial Khovanov homology. This method was generalised by Manolescu for coloured link homlogies ([64], [66]). We would like to mention also the work of Manolescu ([65]) and Gabdaleb, Thiel and Wagner ([25]), where the authors discuss connections between algebraic and geometrical type categorifications for the Jones polynomial.

The Seidel-Smith construction interpolates between the geometry of nilpotent transverse slices for the Lie algebra sl_m and lagrangian Floer theory. They start with a link and see it as a closure of a braid $\beta_m \in B_m$. They construct a certain nilpotent slice inside the Lie algebra sl_{2m} which leads to a symplectic fibration over the configuration space of $2m$ points in the plane. The second step is to define a Lagrangian in the fiber using the theory of vanishing cycles and using a parallel transport method. Then, the monodromy along $\beta_m \cup I^m$ (as a loop in this configuration space) give rise to a second Lagrangian into the fiber. The categorification is then constructed by a lagrangian Floer model, between the initial Lagrangian and the one obtained by monodromy along the braid.

It would be interesting to compare the categorification coming from the topological model from (1.7.0.1, 7) with the Seidel-Smith symplectic categorification for the case of the original Jones polynomial. We remark that our model is based on the interpretation of the braid group as the mapping class group of the punctured disk, and uses this action in a Floer type construction to obtain a categorification, whereas the Seidel-Smith theory uses the braid group as the fundamental group of a configuration space and uses it like a monodromy action.

Concerning the coloured Alexander polynomials, as we have seen, they recover at the first level the original Alexander polynomial. In the same spirit, it would be a good question to investigate if there are some patterns between the categorification from (9) for the Alexander polynomial and Heegaard Floer homology.

Bibliography

[1] B. Abdesselamdag, D. Arnaudonddag and M. Bauerddag - Centre and representations of $U_q(\mathfrak{sl}(2|1))$ at roots of unity. Journal of Physics A: Mathematical and General, Volume 30, Number 3

[2] M. Abouzaid, Ivan Smith- The symplectic arc algebra is formal, Duke Math. J. Volume 165, Number 6 (2016), 985-1060.

[3] M. Abouzaid, I. Smith- Khovanov homology from Floer cohomology Preprint (2015) arXiv:1504.01230

[4] M. Aigner-A Course in Enumeration, Graduate Texts in Mathematics,Volume 238, 2007

[5] Y. Akustu, T. Deguchi, T. Ohtsuki - Invariants of colored links, J. Knot Theory Ramifications 1 161–184, 1992.

[6] C. A. M. Anghel- A Homological model for the coloured Jones polynomials, https://arxiv.org/abs/1712.04873

[7] C. A. M. Anghel- A combinatorial description of the centralizer algebras connected to the Links-Gould Invariant, https://arxiv.org/abs/1712.04878

[8] C. A. M. Anghel, N. Geer- Modified Turaev-Viro Invariants from quantum $sl(2|1)$ https://arxiv.org/abs/1705.03859

[9] J. Barrett, B. Westbury - Spherical categories. Adv. Math. 143 (1999), 357–375.

[10] J. Barrett, B. Westbury - Invariants of piecewise-linear 3-manifolds. Trans. Amer. Math. Soc. 348 (1996), no. 10, 3997–4022.

[11] S. Baseilhac and R. Benedetti - Quantum hyperbolic invariants of 3-manifolds with PSL$(2, \mathbb{C})$-characters. Topology 43 (2004), no. 6, 1373–1423.

[12] A. Beliakova, C. Blanchet -Skein construction of idempotents in the Birman-Murakami-Wenzl algebras . Math Ann 321 (2001) 2, 347-373.

[13] A. Beliakova, S. Wehrli -Categorification of the colored Jones polynomial and Rasmussen invariant of links. Canad. J. Math., 60 1240-1266, 2008

[14] J. Bichon - Cosovereign Hopf algebras, J. Pure Appl. Algebra 157, No.2-3 (2001), 121-133.

[15] Stephen Bigelow- Braid groups are linear, J. Amer. Math. Soc. 14 (2001), 471-486.

[16] Stephen Bigelow - A homological definition of the Jones polynomial. In Invariants of knots and 3-manifolds (Kyoto, 2001), volume 4 of Geom. Topol. Monogr., pages 29-41. Geom. Topol. Publ., Coventry, 2002.

[17] Stephen Bigelow- Homological representations of the Iwahori-Hecke algebra, Geometry and Topology Monographs, Volume 7: Proceedings of the Casson Fest, Pages 493-507

[18] Stephen Bigelow-A homological definition of the HOMFLY polynomial, Algebraic & Geometric Topology 7 (2007) 1409-1440

[19] S. Bigelow, A. Cattabriga, V. Florens- Alexander representation of tangles, Acta Mathematica Vietnamica -Vol. 40, No. 2 (2015)

[20] J. Birman, H. Wenzl- Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc., 313:249–273, 1989.

[21] C. Blanchet, F. Costantino, N. Geer and B. Patureau-Mirand - Non-semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants. Advances in Mathematics 301 (2016), 1–78.

[22] G. E. Bredon- Cosheaves and Homology, Pacific Journal of Mathematics, Vol. 25, No. 1, (1968)

[23] F. Costantino, N. Geer, B. Patureau-Mirand - Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories, Journal of Topology (2014) 7 (4) 1005–1053.

[24] D. De Wit, L.H. Kauffman, J. Links-On the Links-Gould invariant of links, J. Knot Theory Ramifications. 8 (1999), no.2, 165-199. MR 2000j:57020

[25] A. Gadbled, A-L. Thiel, E. Wagner -Categorical action of the extended braid group of affine type A, Commun. Contemp. Math. 19 (2017), no. 3, 1650024, 39 pp.math/1504.07596.

[26] N. Geer, J. Kujawa and B. Patureau-Mirand - Generalized trace and dimension functions on ribbon categories. Selecta Mathematica 17, Issue 2 (2011), Page 453–504.

[27] N. Geer, J. Kujawa and B. Patureau-Mirand - Ambidextrous objects and trace functions for nonsemisimple categories, accepted for publication in Proceedings of the American Mathematical Society, arXiv:1106.4477.

[28] N. Geer, B. Patureau-Mirand, Multivariable link invariants arising from sl(2,1) and the Alexander polynomial, Journal of Pure and Applied Algebra , 210 (2007), no. 1, 283-298.

[29] N. Geer, B. Patureau-Mirand,-On the colored HOMFLY-PT, multivariable and Kashaev link invariants Communications in Contemporary Mathematics Vol. 10, Suppl. 1 (2008) 993–1011

[30] N. Geer, B. Patureau-Mirand, -Multivariable link invariants arising from $\mathfrak{sl}(2|1)$ and the Alexander polynomial, J. Pure Appl. Algebra 210 (2007), no. 1, 283–298.

[31] N. Geer, B. Patureau-Mirand - Topological invariants from non-restricted quantum groups. Algebraic & Geometric Topology, 13 (2013), no. 6, 3305-3363.

[32] N. Geer, B. Patureau-Mirand - Topological invariants from non-restricted quantum groups. Algebraic & Geometric Topology 13 (2013) 3305–3363.

[33] N. Geer, B. Patureau-Mirand, V. Turaev - Modified quantum dimensions and re-normalized link invariants. Compos. Math. 145 (2009), no. 1, 196–212.

[34] N. Geer, B. Patureau-Mirand, V. Turaev - Modified 6j-Symbols and 3–manifold invariants. Advances in Mathematics, Volume 228, Issue 2, (2011), 1163–1202.

[35] N. Geer, B. Patureau-Mirand and A. Virelizier - Traces on ideals in pivotal categories. Quantum Topology, 4 (2013), no. 1, 91–124.

[36] N. P. Ha - Topological invariants from quantum group $\mathcal{U}_q\mathfrak{sl}(2|1)$ at root of unity, preprint, arXiv:1607.03728.

[37] K. Habiro - A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres. Invent. Math. 171 (2008), no. 1, 1–81.

[38] K. Habiro and Thang Le - Unified quantum invariants for integral homology spheres associated with simple Lie algebras. Geom. Topol. 20 (2016), no. 5, 2687–2835.

[39] A. Ishii-The Links–Gould invariant of closed 3-braids, J. Knot Theory Ramifications 13 (2004) 41–56.

[40] A. Ishii - The Links-Gould Polynomial as a Generalization of the Alexander-Conway Polynomial, Pacific J. Math. 225 (2006), 273–285.

[41] Tetsuya Ito - Reading the dual Garside length of braids from homological and quantum representations Comm. Math. Phys., 335(1):345-367, 2015.

[42] Tetsuya Ito- Topological formula of the loop expansion of the colored Jones polynomials, arXiv:1411.5418v1.

[43] Tetsuya Ito -A homological representation formula of colored Alexander invariants Adv. Math. 289, 142-160, 2016.

[44] C. Jackson, T. Kerler- The Lawrence–Krammer–Bigelow representations of the braid groups via $U_q(sl_2)$, Adv. Math. 228, 1689-1717, 2011.

[45] R. M. Kashaev, A link invariant from quantum dilogarithm, Modern Phys. Lett. A 10 (1995), no. 19, 1409-1418.

[46] R. M. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm, Lett. Math. Phys. 39 (1997), no. 3, 269-275.

[47] R. Kashaev - Quantum dilogarithm as a $6j$-symbol. Modern Phys. Lett. A 9 (1994), no. 40, 3757–3768.

[48] C. Kassel, Quantum groups, Springer (1995).

[49] C. Kassel, V. Turaev, Braid groups, Springer, (2008).

[50] S. M. Khoroshkin and V. Tolstoy - Universal R-matrix for quantized (super)algebras. Comm. Math. Phys. 141 (1991), no. 3, 599–617.

[51] M. Khovanov- Categorifications of the colored Jones polynomial. J. Knot Theory Ramific. 14 111–130, 2005.

[52] M. Khovanov, P. Seidel- Quivers, Floer cohomology, and braid group actions J. Amer. Math. Soc., 15:203–271, 2002.

[53] A. N. Kirillov, N. Y. Reshetikhin - Representation algebra $U_q(SL(2))$, q-orthogonal polynomials and invariants of links, LOMI preprint, 1989.

[54] B.M Kohli-On the Links–Gould invariant and the square of the Alexander polynomial , Journal of Knot Theory and Its Ramifications Vol. 25, No. 2 (2016)

[55] D. Krammer- The braid group B4 is linear, Invent. Math. 142 (2000) 451-486

[56] D. Krammer- Braid groups are linear, Ann. of Math. (2) 155 (2002) 131-156

[57] Toshitake Kohno - Homological representations of braid groups and KZ connections. J. Singul., 594-108, 2012.

[58] Toshitake Kohno- Quantum and homological representations of braid groups. Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355-372.

[59] R. J. Lawrence- Homological representations of the Hecke algebra, Comm. Math. Phys. 135 (1990) 141-191

[60] R. J. Lawrence - A functorial approach to the one-variable Jones polynomial. J. Differential Geom., 37(3):689-710, 1993.

[61] Thang Le - Quantum invariants of 3-manifolds: integrality, splitting, and perturbative expansion. Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999). Topology Appl. 127 (2003), no. 1-2, 125-152.

[62] G. Lehrer, R. Zhang- Strongly multiplicity free modules for Lie algebras and quantum groups , J. Algebra 306 (2006), 138-174.

[63] J.R. Links, M.D. Gould- Two variable link polynomials from quantum supergroups, Lett. Math. Phys. 26 (1992) 187–198.

[64] C. Manolescu- Link homology theories from symplectic geometry, Advances in Mathematics, Vol. 211, 363-416, 2007

[65] C. Manolescu- Nilpotent slices, Hilbert schemes, and the Jones polynomial , Duke Mathematical Journal, Vol. 132, 311-369, 2006

[66] C. Manolescu - A link invariant from the symplectic geometry of nilpotent slices.

[67] I. Marin, E. Wagner- A cubic defining algebra for the Links-Gould polynomial, Adv. Math. 248 (2013), 1332-1365

[68] J. Murakami-The Kaufman polynomial of links and representation theory Osaka J. Math., 24:745– 758, 1987.

[69] H. Murakami, J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186 (2001), no. 1, 85-104.

[70] T. Ohtsuki - A polynomial invariant of integral homology 3-spheres. Math. Proc. Cambridge Philos. Soc. 117 (1995), no. 1, 83–112.

[71] P. Ozsvath, Z. Szabo- Holomorphic disks and 3-manifold invariants: properties and applications, Ann. of Math. 159, no. 3, 1159–1245, 2004.

[72] P. Ozsvath, Z. Szabo- Holomorphic disks and topological invariants for closed three-manifolds , Ann. of Math. 159, no. 3, 1027–1158, 2004

[73] U. Pachner - Bistellare quivalenz kombinatorischer Mannigfaltigkeiten. Arch. Math. 30 (1978), no. 1, 89–98.

[74] A. Ram, H. Wenzl-Matrix units for centralizer algebras, J. of Algebra. Vol. 145 (1992), 378–395.

[75] J. Rasmussen-Khovanov homology and the slice genus, Inventiones mathematicae, November 2010, Volume 182, Issue 2, pp 419–447

[76] J. Rasmussen- Floer homology and knot complements, Ph.D. thesis, Harvard University, 2003, math.GT/0306378, 2003.

[77] N. Reshetikhin, V.G. Turaev - Invariants of 3–manifolds via link polynomials and quantum groups. Invent. Math. 103 (1991), no. 3, 547–597.

[78] P. Seidel, I. Smith- A link invariant from the symplectic geometry of nilpotent slices, Duke Math. J.134:453-514, 2006.

[79] P. Seidel, R. Thomas- Braid group actions on derived categories of coherent sheaves, Duke Math. J., 108:37–108, 2001.

[80] R. Stanley- Enumerative combinatorics, Volume 2, Cambridge University Press, 1999

[81] V. Turaev - Quantum invariants of knots and 3-manifolds. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, (1994).

[82] V. Turaev, O. Viro - State sum invariants of 3-manifolds and quantum $6j$-symbols. Topology 31 (1992), no. 4, 865–902.

[83] H. Wenzl-On the structure of Brauer's centralizer algebras, Ann. of Math. 128 (1988), 179-193.

[84] H Yamane - Quantized enveloping algebras associated with simple Lie superalgebras and their universal R-matrices, Publ. Res. Inst. Math. Sci. 30 (1994), no. 1, 15–87.